

AN INVERSE APPROACH TO THE CENTER-FOCI PROBLEM

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ABSTRACT. The classical Center-Focus Problem posed by H. Poincaré in 1880's is concerned on the characterization of planar polynomial vector fields $\mathcal{X} = (-y + X(x, y))\frac{\partial}{\partial x} + (x + Y(x, y))\frac{\partial}{\partial y}$, with $X(0, 0) = Y(0, 0) = 0$, such that all their integral trajectories are closed curves whose interiors contain a fixed point called center or such that all their integral trajectories are spirals called foci. In this paper we state and study the inverse problem to the Center-Foci Problem i.e., we require to determine the analytic planar vector fields \mathcal{X} in such a way that for a given Liapunov function

$$V = V(x, y) = \frac{\lambda}{2}(x^2 + y^2) + \sum_{j=3}^{\infty} H_j(x, y),$$

where $H_j = H_j(x, y)$ are homogenous polynomial of degree j , the following equation holds

$$\mathcal{X}(V) = \sum_{j=3}^{\infty} V_j(x^2 + y^2)^{j+1},$$

where V_j for $j \in \mathbb{N}$ are the Liapunov constants. In particular we study the case when the origin is a center and the vector field is polynomial.

1. INTRODUCTION

The *Center-Focus problem* first was studied by Poincaré [22] and further developed by Lyapunov [19], Bendixson [5] and Frommer [12].

Consider the set Σ of all planar real polynomial vector fields

$$\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y},$$

associated to the differential polynomial systems

$$(1) \quad \dot{x} = P(x, y), \quad \dot{y} = Q(x, y).$$

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here the dot denotes derivative respect to the time t , where $P = P(x, y)$ and $Q = Q(x, y)$ are real coprime in $\mathbb{R}[x, y]$ polynomials of degree $n = \max \{\deg P, \deg Q\}$, in the variables x and y , or more generally, real analytic functions defined in an open neighborhood of the origin $O := (0, 0)$. One assumes that origin is a singular (equilibrium, fixed) point, i.e. $P(0, 0) = Q(0, 0) = 0$.

The equilibrium point is called a center if there exists an open neighborhood U of O that does not contain another equilibrium points such that any trajectory of (1) that intersects $U \setminus O$ in some point is closed. The problem of the center can be formulated as follows: Determine within the class of real planar polynomial vector fields \mathcal{X} , all the systems possessing a center at the origin, i.e. the singular point surrounded by close phase curves.

Suppose That \mathcal{X} is an analytic planar vector field associated to differential system

$$(2) \quad \begin{aligned} \dot{x} &= ax + by + \sum_{k=2}^{\infty} X_k(x, y), \\ \dot{y} &= cx + dy + \sum_{k=2}^{\infty} Y_k(x, y), \end{aligned}$$

where X_k, Y_k are real homogeneous polynomials of degree k . In the aforementioned papers the case of a non-degenerated equilibrium point was studied, i.e. the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

assumed to be invertible. It was proved by Poincaré that a necessary condition for O to be a center is that A has pure imaginary eigenvalues. In this case, making a linear change of variables and then a linear reparametrization of trajectories one reduces (2) to an equivalent system:

$$(3) \quad \dot{x} = -\lambda y + X(x, y), \quad \dot{y} = \lambda x + Y(x, y)$$

where λ is a constant, X and Y are real analytic functions in an open neighborhood of O whose Taylor expansions at O do not contain constant and linear terms. For X, Y polynomials of a given degree, the classical Poincaré Center-Focus Problem asks about conditions on the coefficients of X and Y under which all trajectories of (3) situated in a small open neighborhood of the origin are closed. Poincaré's theorem says that a necessary and sufficient condition to have a center at a singular point at the origin with pure imaginary eigenvalues is that

there exists a local analytic non-constant first integral in the neighborhood of O . The search for an analytic first integral led Poincaré to an algorithm for computing the so-called Poincaré–Lyapunov quantities V_k (*Poincaré criteria*), for $k \in \mathbb{N}$ of the singularity, which are polynomials over \mathbb{Q} in the coefficients of the system. A necessary and sufficient condition to have a center is then the annihilation of all these quantities. In view of Hilbert's basis theorem for this to occur it suffices to have for a finite number of k , $k < j$ and j sufficiently large, $V_k = 0$. Unfortunately, trying to apply Poincaré criteria in the converse direction, gives rise, in the general case, to almost insurmountable difficulties. Therefore, as it was emphasized by Poincaré, it is the matter of great importance to find typical situations for which all equations of the system determining center are satisfied. As an example, Poincaré described a case related to certain symmetries for X and Y in (3).

Although we have an algorithm for computing the Poincaré–Lyapunov constants for singularities with pure imaginary eigenvalues, we have no algorithm to determine how many of them need to be zero to imply that all of them are zero for cubic or higher degree polynomial differential systems. Bautin showed in 1939 that for a quadratic polynomial system, to annihilate all V_k 's it suffices to have $V_k = 0$ for $i = 1, 2, 3$. So the problem of the center is solved for $n = 2$. This problem was solved for the so called quasihomogenous cubic differential systems (see for instance [3, 35, 34, 25, 32]).

$$\begin{aligned}\dot{x} &= -y + a_{20}x^2 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ \dot{y} &= x + b_{20}x^2 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3.\end{aligned}$$

Another necessary and sufficient condition for O to be a center for analytic vector fields was obtained by Lyapunov in [19].

Theorem 1. *Origin is a center for (3) if and only if there exist a first integral $1/2(x^2 + y^2) + f(x, y) = C$, where $f = f(x, y)$ is a real analytic function in an open neighborhood of O whose Taylor expansions at O do not contain constant, linear and quadratic terms.*

Unfortunately, trying to apply the Liapunov's Theorem gives rise, in general, to difficulties comparable with those for Poincaré's criteria.

To show that the singular point is a center we have two basic mechanisms: we either show that the system is symmetric or using Poincaré–Liapunov's Theorems we show that we have a local analytic first integral.

Darboux gave his geometric method of integration in his seminar work [9] of 1878. This method has not played a significant role in the problem of the center until almost the last 20 years of the 20th century when the situation abruptly changed and during the past years the impact of the geometric work of Darboux has been growing steadily in research work on polynomial vector fields. The geometric method of Darboux uses algebraic invariant curves of a system to prove integrability and Darboux's theorem affirms that if we have a sufficient number of such invariant algebraic curves then the system had a first integral which is analytic on the complement of the union of these curves. This connection between the problem of the center and algebraic geometry has come to the forefront in recent years. There were numerous publications on the problem of the center during the last part of the 20th century and the beginning of the 21st century (see for instance [37, 16]).

2. METHODS TO STUDY THE STABILITY OF THE PLANAR NON DEGENERATE DIFFERENTIAL SYSTEM

For simplicity we shall assume that all the functions which appear in this section and in the following ones are of class \mathcal{C}^∞ , although most of the results remain valid under weaker hypotheses.

We shall study the stability of the origin of the planar differential system of the type (3). For this differential system it is possible to determine the general structure of the trajectories in the neighborhood of the origin. We shall study three methods (see for instance [26]).

Method I

Passing to polar coordinates $x = r \cos \vartheta$, $y = r \sin \vartheta$, differential system (3) becomes

$$\begin{aligned}
 \dot{r} &= \cos \vartheta X(r \cos \vartheta, r \sin \vartheta) + \sin \vartheta Y(r \cos \vartheta, r \sin \vartheta) \\
 &:= rR(r, \vartheta), \\
 (4) \quad \dot{\vartheta} &= \frac{1}{r} (\cos \vartheta Y(r \cos \vartheta, r \sin \vartheta) - \sin \vartheta X(r \cos \vartheta, r \sin \vartheta)) + \lambda \\
 &:= \lambda + \theta(r, \vartheta).
 \end{aligned}$$

where $R = R(r, \vartheta)$ and $\theta = \theta(r, \vartheta)$ are real analytic functions in an open neighborhood of the origin whose Taylor expansions at \mathbf{O} do not contain constant and linear terms which converges for r small and vanished for $r = 0$. The coefficients of the Taylor extension are trigonometric polynomials, consequently are 2π periodical functions on ϑ . To

obtain the trajectories from (4) we exclude the variable t . Thus we obtain

$$(5) \quad \frac{dr}{d\vartheta} = \frac{rR}{\lambda + \theta} = r^2 R_2(\vartheta) + r^3 R_3(\vartheta) + \dots$$

The series of the right hand converges for small r . The coefficients $R_j = R_j(\vartheta)$ are polynomial on the variables $\cos \vartheta$ and $\sin \vartheta$. Equation (5) has the trivial solution $r = 0$.

By considering that the right hand of (5) is analytic then any solution $r = r(\vartheta, c)$ of this equation with the initial condition $r(\vartheta, c)|_{\vartheta=0} = c$, can be expand in Taylor series as follows

$$(6) \quad r(\vartheta, c) = r_1(\vartheta)c + r_2(\vartheta)c^2 + \dots$$

with converges for c small and satisfy the initial conditions

$$(7) \quad r_1(0) = 1, \quad r_2(0) = r_3(0) = \dots = 0.$$

By inserting (7) into (5) we obtain that

$$(8) \quad \frac{dr_1(\vartheta)}{d\vartheta} = 0, \quad \frac{dr_j(\vartheta)}{d\vartheta} = F_j(\vartheta),$$

for $j = 2, 3, \dots$, where $F_2 = r_1^2 R_2$, $F_3 = r_1^3 R_3 + 2r_1 r_2 R_2$, \dots

After integration (8) and in view of (7) we obtain

$$r_1 = 1, \quad r_j = \int_0^\vartheta F_j(\vartheta) d\vartheta,$$

for $j = 2, 3, \dots$

By considering that the function R_2 is 2π periodical functions then we have the following expansion in Fourier's series

$$R_2(\vartheta) = g_2 + \sum_{n=1}^{\infty} \left(A_n \cos \left(\frac{2\pi n \vartheta}{\omega} \right) + B_n \sin \left(\frac{2\pi n \vartheta}{\omega} \right) \right),$$

Consequently

$$r_2 = \int_0^\vartheta R_2(\vartheta) d\vartheta = g_2 \vartheta + \varphi_2(\vartheta), \quad \varphi_2(\vartheta + 2\pi) = \varphi_2(\vartheta).$$

If $g_2 = 0$ then r_2 is a 2π periodical function. Hence $F_3 = R_3 + 2r_2 R_2$ is 2π periodical function. Thus have the following expansion in Fourier's series

$$F_3(\vartheta) = \tilde{g} + \sum_{n=1}^{\infty} \left(\tilde{A}_n \cos \left(\frac{2\pi n \vartheta}{\omega} \right) + \tilde{B}_n \sin \left(\frac{2\pi n \vartheta}{\omega} \right) \right).$$

Hence

$$r_3 = \int_0^\vartheta F_3(\vartheta) d\vartheta = \tilde{g} \vartheta + \tilde{\varphi}(\vartheta), \quad \tilde{\varphi}(\vartheta + 2\pi) = \tilde{\varphi}(\vartheta).$$

If r_m is the first non-periodical function then it takes the form

$$(9) \quad r_m(\vartheta) = \int_0^\vartheta F_m(\vartheta) d\vartheta = g \vartheta + \varphi(\vartheta), \quad \varphi(\vartheta + 2\pi) = \varphi(\vartheta)$$

If all coefficients r_m are periodic functions then (7) is a periodic function for all values of c which belong to the convergence region of the series (7). Consequently all the integral curves which are in the neighborhood of the origin are closed curves which contain the point $(0, 0)$. Poincaré called this point *center*. The origin in this case is stable.

Now we assume that r_2, r_3, \dots, r_{m-1} are periodic functions and r_m can be determine by the formula (9) with $g \neq 0$.

We shall transform equation (5) by using the change

$$\begin{aligned} r &= \varrho + \varrho^2 r_2(\vartheta) + \dots + \varrho^{m-1} r_{m-1}(\vartheta) + \varrho^m \varphi(\vartheta) \\ &= \varrho + \varrho^2 r_2(\vartheta) + \dots + \varrho^{m-1} r_{m-1}(\vartheta) + \varrho^m (r_m(\vartheta) - g\vartheta). \end{aligned}$$

Hence in view of (8) we get

$$\begin{aligned} \frac{d\varrho}{d\vartheta} &= \frac{\varrho^2 F_2 + \varrho^3 F_3 + \dots + \varrho^m F_m + \varrho^{m+1} F_{m+1} + \dots}{1 + 2r_2\varrho + \dots + m\varrho^{m-1}\varphi(\vartheta)} \\ &= \frac{g\varrho^m + \varrho^{m+1} F_{m+1} + \dots}{1 + 2r_2\varrho + \dots + m\varrho^{m-1}\varphi(\vartheta)} = g\varrho^m + \varrho^{m+1} \tilde{F}_{m+1}(\vartheta) + \dots \end{aligned}$$

From this relation follows that in the small neighborhood of the origin $\frac{d\varrho}{d\vartheta}$ has constant sign which coincide with the sign of g . If $g < 0$ then if $\vartheta \rightarrow +\infty$ then $\varrho(\vartheta) \rightarrow 0$, if $g > 0$ then when $\vartheta \rightarrow -\infty$ then $\varrho(\vartheta) \rightarrow 0$ the same behavior has the radius r . Consequently all integral curves which are in the vicinity of the origin are spirals. Poincaré called this trajectory *foci*. If $g < 0$ is asymptotic stable foci and if $g > 0$ is unstable foci.

The problems arise when the first non periodic coefficient has a big index, in this case it is necessary to do a lot of tedious computations. But the most complicated problem appears when all the coefficients r_j are periodic functions, ie. when the origin is a center. In this case we have an infinity number of conditions and it is practically impossible to show that all these conditions hold. In the general case this problem is open. Poincaré in [22] proved the following theorem

Theorem 2. *Polynomial differential system*

$$\dot{x} = -y + \sum_{j=2}^m X_j(x, y), \quad \dot{y} = x + \sum_{j=2}^m Y_j(x, y),$$

has a center at the origin if and only if there exists a local first integral of the form

$$F(x, y) = \frac{1}{2}(x^2 + y^2) + f(x, y),$$

defined in a neighborhood of the origin, where $f = f(x, y)$ is a real analytic functions in an open neighborhood of \mathbf{O} whose Taylor expansions at \mathbf{O} do not contain constant, linear and quadratic terms.

If we determine the "time" dependence of the variables x and y i.e, $x = x(t)$ and $y = y(t)$ then these functions are periodic functions with period T which depend analytically on the parameter c . To obtain this dependence we transform the second of equation (4) by using (6). We obtain

$$\frac{d\vartheta}{dt} = \lambda + \theta(r, \vartheta) = \lambda (1 + c\theta_1(\vartheta) + c^2\theta_2(\vartheta) + \dots),$$

where $\theta_j = \theta_j(\vartheta)$ are convenient periodic function. Without loss the generality we assume that $\vartheta|_{t=0} = 0$. Thus we have that

$$\begin{aligned} t(\vartheta) &= \frac{1}{\lambda} \int_0^{\vartheta} \frac{d\vartheta}{1 + c\theta_1(\vartheta) + c^2\theta_2(\vartheta) + \dots} \\ &= \frac{1}{\lambda} \int_0^{\vartheta} (1 + c\tilde{\theta}_1(\vartheta) + c^2\tilde{\theta}_2(\vartheta) + \dots), \end{aligned}$$

in view of periodic character of function θ_j we deduce the relation

$$t(\vartheta + 2\pi) - t(\vartheta) = \frac{1}{\lambda} \int_0^{\vartheta+2\pi} \frac{d\vartheta}{1 + c\theta_1(\vartheta) + c^2\theta_2(\vartheta) + \dots}.$$

Thus we obtain that if the origin is center then periodic solution has period T such that

$$\begin{aligned} T := t(2\pi) - t(0) &= \frac{1}{\lambda} \int_0^{2\pi} \frac{d\vartheta}{1 + c\theta_1(\vartheta) + c^2\theta_2(\vartheta) + \dots} \\ &= \frac{2\pi}{\lambda} (1 + h_1c + h_2c^2 + \dots), \end{aligned}$$

where $h_j = \frac{1}{2\pi} \int_0^{2\pi} \theta_j(\vartheta) d\vartheta$. This series converge for small c . The analytic function $T = T(c)$ is called *period function*. If $T(c) = \frac{2\pi}{\lambda}$ then the center is called *isochronous center*.

Method II

To study the case when the origin is a center for (3) we transform this equation with the change

$$t = \frac{\tau}{\lambda} (1 + h_1 c + h_2 c^2 + \dots).$$

Under this change (3) becomes

$$\begin{aligned} x' &= \left(-y + \frac{X}{\lambda}\right) (1 + h_1 c + h_2 c^2 + \dots) \\ &= \left(-y + \frac{1}{\lambda} (X_2 + X_3 + \dots)\right) (1 + h_1 c + h_2 c^2 + \dots), \\ (10) \quad y' &= \left(x + \frac{Y}{\lambda}\right) (1 + h_1 c + h_2 c^2 + \dots) \\ &\quad \left(x + \frac{1}{\lambda} (Y_2 + Y_3 + \dots)\right) (1 + h_1 c + h_2 c^2 + \dots), \end{aligned}$$

where $' = \frac{d}{d\tau}$, and $X_k = X_k(x, y)$, $Y = Y_k(x, y)$ are homogenous polynomial of degree k , for $k = 2, 3, \dots$. Solution of these equations with the initial conditions

$$x|_{\tau=0} = c, \quad y|_{\tau=0} = 0,$$

is 2π periodic function. By considering that (10) has an analytic dependence on the parameter c . In view of the well known theorem of the dependence of the solutions on the parameter we deduce the validity of the relations

$$(11) \quad \begin{aligned} x &= x(\tau) = cx_1(\tau) + c^2 x_2(\tau) + \dots, \\ y &= y(\tau) = cy_1(\tau) + c^2 y_2(\tau) + \dots, \end{aligned}$$

these series converges for small values of c . Clearly that these solutions are such that

$$x(\tau + 2\pi) = x(\tau), \quad y(\tau + 2\pi) = y(\tau),$$

with the initial conditions

$$(12) \quad x_1|_{\tau=0} = 1, \quad x_2|_{\tau=0} = 0, \dots, \quad y_1|_{\tau=0} = 0, \quad y_2|_{\tau=0} = 0, \dots$$

Inserting (11) into (10) and comparing different power coefficients of c we obtain the following set of differential equations with respect to x_k

and y_k .

$$\begin{aligned} x'_1 &= -y_1, & y'_1 &= x_1, \\ x'_2 &= -y_2 - h_1 y_1 + \frac{1}{\lambda} X_2(\cos \tau, \sin \tau), & y'_2 &= -x_2 - h_1 x_1 + \frac{1}{\lambda} Y_2(\cos \tau, \sin \tau), \\ &\vdots & &\vdots \\ x'_k &= -y_k - h_{k-1} \sin \tau + P_k, & y'_k &= -x_k - h_{k-1} \cos \tau + Q_k, \\ &\vdots & &\vdots \end{aligned}$$

where P_k and Q_k are polynomials with respect to $x_2, y_2, \dots, x_{k-1}, y_{k-1}$ which coefficients which depends on the parameters h_1, h_2, \dots, h_{k-1} .

In view of that the first two equations with the initial conditions $x_1(0) = 1$ and $y_1(0) = 0$ admits the solutions

$$x_1(\tau) = \cos \tau, \quad y_1(\tau) = \sin \tau,$$

then we obtain that the following two differential equations become

$$(13) \quad x'_2 = -y_2 + P_2(\tau), \quad y'_2 = x_2 + Q_2(\tau),$$

where $P_2(\tau)$ and $Q_2(\tau)$ are 2π periodic functions such that

$$P_2(\tau) = -h_1 \sin \tau + \frac{1}{\lambda} X_2(\cos \tau, \sin \tau), \quad Q_2(\tau) = h_1 \cos \tau + \frac{1}{\lambda} Y_2(\cos \tau, \sin \tau).$$

Differential system (13) can be rewritten as follows

$$z'_2 = iz_2 + (P_2(\tau) + iQ_2(\tau)), \quad z_2 = x_2 + iy_2.$$

The solution of this system by considering that $z_2(0) = 0$, is

$$z_2 = \int_0^\tau e^{i(\tau-\varsigma)} (P_2(\varsigma) + iQ_2(\varsigma)) d\varsigma.$$

Thus this solution is 2π periodic function if and only if

$$(14) \quad \int_0^{2\pi} e^{-i\varsigma} (P_2(\varsigma) + iQ_2(\varsigma)) d\varsigma = 0.$$

By expanding in Fourier's series the functions P_2 and Q_2 i.e.

$$\begin{aligned} P_2 &= a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\tau) + b_n \sin(n\tau)) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\tau}, \\ Q_2 &= b_0 + \sum_{n=1}^{\infty} (A_n \cos(n\tau) + B_n \sin(n\tau)) = \sum_{n=-\infty}^{\infty} \beta_n e^{in\tau}, \end{aligned}$$

In these notations condition (14) takes the form

$$\int_0^{2\pi} e^{i(n-1)\varsigma} (\alpha_n + i\beta_n) d\varsigma = \alpha_1 + i\beta_1 = a_1 + B_1 + i(A_1 - b_1) = 0.$$

If the all functions $x_2, y_2, \dots, x_{k-1}, y_{k-1}$ are periodic functions, then the functions x_k and y_k are periodic if and only if

$$\int_0^{2\pi} e^{-i\varsigma} (P_k(\varsigma) + iQ_k(\varsigma)) d\varsigma = 0,$$

hence

$$\begin{aligned} 2\pi h_{k-1} + \int_0^{2\pi} (P_k(\varsigma) \cos \varsigma - iQ_k(\varsigma) \sin \varsigma) d\varsigma &= 0, \\ \int_0^{2\pi} (P_k(\varsigma) \sin \varsigma - Q_k(\varsigma) \cos \varsigma) d\varsigma &= 0. \end{aligned}$$

The first relation always holds by choosing properly the constant h_{k-1} . If second relation do not takes place, ie.

$$\int_0^{2\pi} (P_k(\varsigma) \sin \varsigma - Q_k(\varsigma) \cos \varsigma) d\varsigma = \frac{a_1 + B_1}{2} \neq 0,$$

consequently we have that

$$\begin{aligned} y|_{\tau=2\pi} &= y|_{\tau=0} = 0, \\ x|_{\tau=2\pi} &= c + 2\pi \frac{a_1 + B_1}{2} c^m + \dots, \quad x|_{\tau=0} = c. \end{aligned}$$

If $a_1 + B_1 > 0$ ($a_1 + B_1 < 0$) then the origin is asymptotically stable (unstable) foci.

Finally we observe that the first constants h_k different from zero has always even index.

Method III

Another method to study the origin's stability by applying the following Liapunov's Theorem.

Consider the differential system

$$(15) \quad \dot{\mathbf{x}} = X(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0,$$

where $\mathbf{x} = (x_1, x_2, \dots, x_N)$, and $X : \mathcal{D} \rightarrow \mathbb{R}^n$ is analytic function, \mathcal{D} is an open set containing the origin, which is an equilibrium point, i.e. $X(0) = 0$.

Lyapunov, in his original 1892 work, proposed two methods for demonstrating stability (see [26, 19]). The first method developed the solution in a series which was then proved convergent within limits.

The second method, which is almost universally used nowadays, makes use of a Lyapunov function $V(\mathbf{x})$ which has an analogy to the potential function of classical dynamics. It is introduced as follows for a system having a point of equilibrium at $\mathbf{x} = \mathbf{0}$. Consider a function $V(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- (i) $V(x) \geq 0$ with equality if and only if $\mathbf{x} = \mathbf{0}$, i.e. V is a positive definite function.
- (ii) $\dot{V}(\mathbf{x}) = \frac{d}{dt}V(\mathbf{x}) \leq 0$ with equality not constrained to only $\mathbf{x} = \mathbf{0}$, i.e. function $\dot{V}(\mathbf{x})$ is negative semi-definite. For asymptotic stability, $\dot{V}(\mathbf{x})$ is required to be negative definite. Then $V(\mathbf{x})$ is called a Lyapunov function candidate and the system is stable in the sense of Lyapunov.

It is easier to visualize this method of analysis by thinking of a physical system (e.g. vibrating spring and mass) and considering the energy of such a system. If the system loses energy over time and the energy is never restored then eventually the system must grind to a stop and reach some final resting state. This final state is called the attractor. However, finding a function that gives the precise energy of a physical system can be difficult, and for abstract mathematical systems, economic systems or biological systems, the concept of energy may not be applicable.

Lyapunov's realization was that stability can be proven without requiring knowledge of the true physical energy, provided a Lyapunov function can be found to satisfy the above conditions.

Theorem 3. (*Liapunov's Theorem*) *Let $\mathbf{x} = \mathbf{0}$ be an equilibrium point for the system (15). Let V be a positive definite continuously differentiable function in $D \subseteq \mathbb{R}^N$.*

- (1) *If $\frac{dV}{dt}$ is negative semi-definite, then the origin is stable.*
- (2) *If $\frac{dV}{dt}$ is negative definite, then the origin is asymptotically stable.*
- (3) *If $\frac{dV}{dt}$ is positive definite, then the origin is unstable.*

Now we apply Liapunov's Theorem to study the stability of the analytic planar vector field associated to differential system (15) with $N = 2$, and $x_1 = x$, $x_2 = y$

$$\dot{x} = -y + \sum_{j=1}^{\infty} X_j(x, y), \quad \dot{y} = x + \sum_{j=1}^{\infty} Y_j(x, y),$$

where X_j and Y_j are homogenous polynomials of degree j .

We will need the following results (see for instance [25]).

Theorem 4. *The partial first order differential equations*

$$x \frac{\partial f_n}{\partial y} - y \frac{\partial f_n}{\partial x} = g_n,$$

where f_n and g_n are homogenous polynomial of degree n , admits a unique solution if n is odd and if n is even i.e. $n = 2m$ and g_n is a polynomial of degree $2m$ and such that

$$g_{2m} = \sum_{k+j=2m} a_{kj} x^j y^k + K_{m-1} (x^2 + y^2)^m.$$

where a_{kj} are constants and K_{m-1} are arbitrary constants such that

$$K_{m-1} = -\frac{1}{2\pi} \sum_{k+j=2m} a_{kj} \int_0^{2\pi} \cos^k t \sin^j t dt.$$

The aim of the following studies is to construct the Liapunov function which satisfy Liapunov's Theorem . Thus we shall determine the function V :

$$V = \sum_{n=2}^{\infty} H_n(x, y) := \frac{1}{2}(x^2 + y^2) + \sum_{n=3}^{\infty} H_n(x, y),$$

where $H_n(x, y)$ are homogenous polynomials of degree n , i.e.

$$x \frac{\partial H_n}{\partial x} + y \frac{\partial H_n}{\partial y} = n H_n.$$

We choose the functions H_n in such a way that $\frac{dV}{dt}$ is a positive (negative) definite function. By considering that

$$\begin{aligned}
 \frac{dV}{dt} &= \left(x + \frac{\partial H_3}{\partial x} + \dots \right) (-y + X_2 + X_3 + \dots) \\
 &\quad + \left(y + \frac{\partial H_3}{\partial y} + \dots \right) (x + Y_2 + Y_3 + \dots) \\
 &= xX_2 + yY_2 + \{H_2, H_3\} \\
 &\quad + xX_3 + yY_3 + \frac{\partial H_3}{\partial x}X_2 + \frac{\partial H_3}{\partial y}Y_2 + \{H_2, H_4\} \\
 &\quad xX_4 + yY_4 + \frac{\partial H_3}{\partial x}X_3 + \frac{\partial H_3}{\partial y}Y_3 \\
 &\quad + \frac{\partial H_4}{\partial x}X_2 + \frac{\partial H_4}{\partial y}Y_2 + \{H_2, H_5\} + \dots \\
 &\quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\
 &\quad xX_m + yY_m + \frac{\partial H_3}{\partial x}X_{m-1} + \frac{\partial H_3}{\partial y}Y_{m-1} + \dots \\
 &\quad + \frac{\partial H_m}{\partial x}X_2 + \frac{\partial H_m}{\partial y}Y_2 + \{H_2, H_{m+1}\} \\
 &\quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\
 &:= L_3 + L_4 + L_5 + \dots + L_{m+1} + \dots
 \end{aligned}
 \tag{16}$$

where $\{f, g\} := \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x}$, L_n are homogenous polynomial of degree n . Function $\frac{dV}{dt}$ will be positive (negative) definite it is necessary that it beginning with even term. Thus

$$L_3 = xX_2 + yY_2 + \{H_2, H_3\} = 0,$$

By considering that X_2, Y_2, H_2 and H_3 are homogenous polynomial, then in view of Theorem 4 we obtain that there exist an unique homogenous polynomial H_3 which is a solution of this equation.

Another hand, from the relation

$$L_4 = xX_3 + yY_3 + \frac{\partial H_3}{\partial x}X_2 + \frac{\partial H_3}{\partial y}Y_2 + \{H_2, H_4\},$$

and by considering that

$$F_4 := -xX_3 - yY_3 - \frac{\partial H_3}{\partial x}X_2 - \frac{\partial H_3}{\partial y}Y_3 + L_4$$

is homogenous polynomial of degree four, then taking $L_4 = V_1(x^2 + y^2)^2$, and in view of Theorem 4 we deduce that there exist an unique polynomial H_4 .

By continue these process we obtain that it is possible to construct the Liapunov function which satisfy Liapunov's Theorems.

Clearly is the constants $V_j = 0$ for $j \in \mathbb{N}$ then there exist a first integral

$$\frac{\lambda}{2}(x^2 + y^2) + \sum_{j=3}^{\infty} H_j = C.$$

Consequently the origin is a center. If there exist a non-zero Liapunov constant V_j then in view of the relation

$$\frac{dV}{dt} = V_j(x^2 + y^2)^{j+1} + \dots,$$

the origin is asymptotically stability foci if $V_j < 0$ and unstability foci if $V_j > 0$.

3. POINCARÉ-LIAPUNOV INTEGRABILITY

We introduce the following definition

Let U be an open and dense set in \mathbb{R}^2 . We say that a non-constant C^1 function $H: U \rightarrow \mathbb{R}$ is a *first integral* of the polynomial vector field \mathcal{X} on U , if $H(x(t), y(t))$ is constant for all values of t for which the solution $(x(t), y(t))$ of \mathcal{X} is defined on U . Clearly H is a first integral of \mathcal{X} on U if and only if $\mathcal{X}H = 0$ on U .

Differential system (1) for which the origin is a singular point, i.e. $P(0, 0) = Q(0, 0) = 0$ we call *Poincaré-Liapunov integrable* if admits a first integral $F = F(x, y)$ such that the development at the point $(0, 0)$ is the following

$$F = F(0, 0) + \frac{1}{2}(ax^2 + by^2 + cxy) + f(x, y),$$

where f is a real analytic functions in an open neighborhood of O whose Taylor expansions at O do not contain constant and linear terms and $ax^2 + by^2 + cxy$ is a definite positive (negative) quadratic form (see [29]).

Now we shall study the case when the differential system is given by the formula (3). Particular cases of Poincaré-Liapunov integrable system are the following:

- (i) System (3) is a Hamiltonian system, i.e.

$$X(x, y) = -\frac{\partial F(x, y)}{\partial y}, \quad Y(x, y) = \frac{\partial F(x, y)}{\partial x}.$$

Hence F is a first integral.

- (ii) Beside Hamiltonian systems there is another class of systems (3) for which the origin is a center, namely that of reversible systems satisfying the following definition.

We say that system (3) is *reversible with respect to the straight line l* through the origin if it is invariant with respect to reversion about l and a reversion of time t , (see for instance [8]).

The following criterion going back to Poincaré, namely (see for instance [24], p.122)

Theorem 5. *The origin is a center of (3) if it is reversible.*

We apply this theorem for the case when (3) is invariant under the transformations $(x, y, t) \rightarrow (-x, y, -t)$ and $(x, y, t) \rightarrow (x, -y, -t)$.

By introducing the complex coordinates $z = x + iy$, $\bar{z} = x - iy$, we can rewrite the analytic planar differential systems as follows

$$(17) \quad \dot{z} = \sum_{n,k=0}^M a_{nk} z^n \bar{z}^k,$$

where $a_{nk} \in \mathbb{C}$, and M can be equal to infinity.

Proposition 6. *Differential systems (17) is invariant under the change $(x, y, t) \rightarrow (x, -y, -t)$, if and only if $a_{nk} = ib_{nk}$ where $b_{nk} \in \mathbb{R}$.*

Proof. Under the change $(t, x, y) \rightarrow (-t, x, -y)$ differential system (17) takes the form

$$\dot{\bar{z}} = - \sum_{n,k=0}^{\infty} a_{nk} \bar{z}^n z^k.$$

On the other hand from (17) follows that

$$\dot{\bar{z}} = \sum_{n,k=1}^{\infty} \bar{a}_{nk} \bar{z}^n z^k$$

hence $\bar{a}_{nk} = -a_{nk}$. From this relation the proof follows. \square

We shall study the case when

$$(18) \quad \dot{z} = i \left(z + \sum_{n,k=2}^{\infty} b_{nk} z^n \bar{z}^k \right), \quad b_{nk} \in \mathbb{R}.$$

Consequently analytic differential system (3) is reversible if and only

$$X(x, y) = yf(x, y^2), \quad Y(x, y) = g(x, y^2),$$

where f and g are analytic functions such that $g(0, 0) = 0$.

From this result follows that differential system

$$(19) \quad \dot{x} = -y + yf(x, y^2), \quad \dot{y} = x + g(x, y^2),$$

admits a center at the origin. In particular we deduce the following system (Lienard's type equation)

$$\dot{x} = -y, \quad \dot{y} = x + g(y^2).$$

By introducing the variable $Y = y^2$ from (19) we obtain the differential equation

$$\frac{dY}{dx} = \frac{x + g(x, Y)}{-1 + f(x, Y)}.$$

Proposition 7. *Differential systems (17) is invariant under the change $(x, y, t) \longrightarrow (-x, y, -t)$, if and only if*

$$a_{nk} = (-1)^{n+k} \bar{a}_{nk}.$$

Proof. Under the change $(x, y, t) \longrightarrow (-x, y, t)$ we obtain that

$$z \longrightarrow -\bar{z}, \quad \bar{z} \longrightarrow -z, \quad t \longrightarrow -t,$$

thus differential system (17) takes the form

$$\dot{\bar{z}} = \sum_{n,k=0}^{\infty} a_{nk} (-1)^{n+k} \bar{z}^k z^n.$$

On the other hand from (17) follows that

$$\dot{\bar{z}} = \sum_{n,k=1}^{\infty} \bar{a}_{nk} \bar{z}^n z^k,$$

consequently $\bar{a}_{nk} = (-1)^{n+k} a_{nk}$. From this relation the proof follows.

Consequently analytic differential system (3) is invariant under the change $(x, y, t) \longrightarrow (-x, y, -t)$, if and only

$$X(x, y) = f(x^2, y), \quad Y(x, y) = xg(x^2, y),$$

where f and g are analytic functions such that $f(0, 0) = 0$.

From this result follows that differential system

$$(20) \quad \dot{x} = -y + f(x^2, y), \quad \dot{y} = x + xg(x^2, y),$$

admits a center at the origin.

Particular case of this system is (Lienard's type equation)

$$\dot{x} = -y + f(x^2), \quad \dot{y} = x,$$

Under the change $X = x^2$ we deduce from (20) the differential equation

$$\frac{dX}{dy} = \frac{-y + f(X, y)}{1 + g(X, y)}.$$

□

Differential system

$$(21) \quad \dot{x} = -y + X_m(x, y), \quad \dot{y} = x + Y_m(x, y),$$

where $X_m = X_m(x, y)$ and $Y = Y_m(x, y)$ are homogenous polynomial of degree m , is called *quasi-homogenous differential system*.

Corollary 8. *The quasihomogenous differential system (21) is invariant under the change $(x, y, t) \longrightarrow (-x, y, -t)$, if and only if*

$$a_{nk} = \begin{cases} a_{nk}^-, & \text{if } m = 2l, \\ -a_{nk}^-, & \text{if } m = 2l + 1. \end{cases}$$

Proof. Indeed from the relations $\bar{a}_{nk} = (-1)^{n+k} a_{nk} = (-1)^m a_{nk}$ the proof follows. □

Below for simplicity shall say that the planar differential system is reversible if it is invariant under the change $(x, y, t) \longrightarrow (x, -y, -t)$, or $(x, y, t) \longrightarrow (-x, y, -t)$.

From Proposition 7 and Corollary 8 we get the following result.

Proposition 9. *Quasihomogenous differential system (21) for $m = 2l + 1$ is reversible if and only if $\Re(a_{jk}) = 0$ for $j + k = 2l + 1$, i.e.*

$$(22) \quad \begin{aligned} \dot{x} &= y (a_{00} + a_{2l,0}x^{2l} + a_{2l-2,2}x^{2l}y^2 + \dots + a_{0,2l}y^{2l}), \\ \dot{y} &= x (b_{00} + b_{2l,0}x^{2l} + b_{2l-2,2}x^{2l}y^2 + \dots + b_{0,2l}y^{2l}), \end{aligned}$$

(iii) The following condition is well known as *weak condition of the center* (see for instance[2]).

Proposition 10. (*Weak condition of the center*) *The origin is a center of (21) if there exist $\mu \in \mathbb{R}$ such that*

$$(23) \quad (x^2 + y^2) \left(\frac{\partial X_m}{\partial x} + \frac{\partial Y_m}{\partial y} \right) = \mu (xX_m + yY_m),$$

and either $m = 2k$ is even, or $m = 2k - 1$ is odd and $\mu \neq 2k$, or $m = 2k - 1$ and

$$(24) \quad \int_0^{2\pi} (xX_m + yY_m) |_{x=\cos t, y=\sin t} dt = 0$$

In [10] the author proved that if $\mu = 2m$ then there exist the rational first integral

$$\frac{x^2 + y^2 - 2(xY_m - yX_m)}{(x^2 + y^2)^m} = \text{Const.}$$

- (iv) Another particular case of differential system with a center is the system which satisfy the Cauchy–Riemann conditions (see for instance [8]).

Proposition 11. *System (3) has a center at the origin if X and Y satisfy the Cauchy-Riemann equation*

$$(25) \quad \frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y}, \quad \frac{\partial X}{\partial y} = -\frac{\partial Y}{\partial x}$$

Center for which (25) holds is called *holomorphic center* which is a particular case of the isochronous center.

The most general analytic planar differential system with holomorphic center is

$$\dot{z} = \sum_{j=1}^{\infty} c_j z^j, \quad c_j \in \mathbb{C}.$$

4. INVERSE PROBLEM OF THE CENTER PROBLEM FOR THE POLYNOMIAL PLANAR VECTOR FIELDS.

In this section we state and solve the following inverse problem for of the center for the polynomial planar vector fields.

Problem 12. *Determine the polynomial planar vector fields*

$$(26) \quad \mathcal{X} = (-y + \sum_{j=2}^m X_j(x, y)) \frac{\partial}{\partial x} + (x + \sum_{j=2}^m Y_j(x, y)) \frac{\partial}{\partial y},$$

where X_j and Y_j for $j = 2, 3, \dots, m$ are unknown homogenous polynomial of degree j , in such a way that the

$$(27) \quad V = \sum_{j=2}^{\infty} H_j(x, y) = \frac{\lambda}{2}(x^2 + y^2) + \sum_{j=3}^{\infty} H_j(x, y) = C,$$

is it a first integral, where H_j is a homogenous polynomial of degree j , for $j = 2, 3, \dots$.

Analogously problem can be stated for the case when the vector field \mathcal{X} is analytic.

The solution of this inverse problem we obtain from the following theorem.

Theorem 13. *The most general polynomial planar vector field of degree n for which (27) is a first integral is*

$$(28) \quad \dot{x} = \sum_{j=2}^{m+1} g_{m+1-j} \{\Psi_j, x\}, \quad \dot{y} = \sum_{j=2}^{m+1} g_{m+1-j} \{\Psi_j, y\},$$

with the complementary conditions

$$(29) \quad \sum_{j=1}^m g_{j-1} \{H_2, H_{n+2-j}\} + \{H_{m+1}, H_{n+2-n}\} = 0, \quad n = m, m+1, \dots$$

where g_{n+1-j} is an arbitrary homogenous polynomial of degree $n+1-j$, $j \geq 0$, and

$$\Psi_j = \sum_{k=2}^j H_k, \quad \text{for } j = 2, \dots, m+1,$$

where H_k are homogenous polynomial of degree k .

Proof. From the Liapunov Theorem 1 we obtain that the origin is center for the vector field \mathcal{X} if and only if (27) is it a first integral. Hence (16) holds. This equation is satisfied if and only if the following relations

take place

$$\begin{aligned}
xX_2 + yY_2 + \{H_2, H_3\} &= 0, \\
xX_3 + yY_3 + \frac{\partial H_3}{\partial x}X_2 + \frac{\partial H_3}{\partial y}Y_2 + \{H_2, H_4\} &= 0, \\
xX_4 + yY_4 + \frac{\partial H_3}{\partial x}X_3 + \frac{\partial H_3}{\partial y}Y_3 + \frac{\partial H_4}{\partial x}X_2 + \frac{\partial H_4}{\partial y}Y_2 + \{H_2, H_5\} &= 0, \\
\vdots & \quad \quad \quad \vdots & \quad \quad \quad \vdots & \quad \quad \quad \vdots & \quad \quad \quad \vdots \\
xX_n + yY_n + \frac{\partial H_3}{\partial y}Y_{n-1} + \dots + \frac{\partial H_n}{\partial x}X_2 + \frac{\partial H_n}{\partial y}Y_2 + \{H_2, H_{n+1}\} &= 0, \\
xX_{n+1} + yY_{n+1} + \frac{\partial H_3}{\partial y}Y_n + \dots + \frac{\partial H_{n+1}}{\partial x}X_2 + \frac{\partial H_{n+1}}{\partial y}Y_2 + \{H_2, H_{n+2}\} &= 0, \\
\vdots & \quad \quad \quad \vdots & \quad \quad \quad \vdots & \quad \quad \quad \vdots & \quad \quad \quad \ddots
\end{aligned}$$

The first equation can be rewritten as follows

$$x \left(X_2 + \frac{\partial H_3}{\partial y} \right) + y \left(Y_2 - \frac{\partial H_3}{\partial x} \right) = 0,$$

by solving with respect to X_2 and Y_2 we obtain that

$$\begin{aligned}
X_2 &= -\frac{\partial H_3}{\partial y} - yg_1 = \{H_3, x\} + g_1\{H_2, x\}, \\
Y_2 &= \frac{\partial H_3}{\partial x} + xg_1 = \{H_3, y\} + g_1\{H_2, y\},
\end{aligned}$$

where $g_1 = g_1(x, y)$ is an arbitrary homogenous polynomial of degree one. Inserting these polynomial into the second equation we obtain

$$x \left(X_3 - \frac{\partial H_4}{\partial y} + g_1 \frac{\partial H_3}{\partial y} \right) + y \left(Y_3 - \frac{\partial H_4}{\partial x} - g_1 \frac{\partial H_3}{\partial x} \right) = 0,$$

solving this equation with respect to X_3 and Y_3 we have

$$\begin{aligned}
X_3 &= -\frac{\partial H_4}{\partial y} - g_1 \frac{\partial H_3}{\partial y} - yg_2 = \{H_4, x\} + g_1\{H_3, x\} + g_2\{H_2, x\}, \\
Y_3 &= \frac{\partial H_4}{\partial x} + g_1 \frac{\partial H_3}{\partial x} + xg_2 = \{H_4, y\} + g_1\{H_3, y\} + g_2\{H_2, y\},
\end{aligned}$$

where $g_2 = g_2(x, y)$ is an arbitrary homogenous polynomial of degree two. By continue this process we obtain $X_4, Y_4, \dots, X_n, Y_n$. Inserting the obtained polynomials in the remaining partial differential equations

we deduce (29), and introducing the respectively notations we get

$$\begin{aligned} X_2 + X_3 + \dots + X_m &= \sum_{j=2}^{m+1} g_{m+1-j} \{\Psi_j, x\}, \\ Y_2 + Y_3 + \dots + Y_m &= \sum_{j=2}^{m+1} g_{m+1-j} \{\Psi_j, y\}. \end{aligned}$$

By inserting X_j and Y_j for $j = 1, \dots, m$ into the remain equations we deduce the partial differential equations (29). Thus the proof of the theorem follows. \square

From Theorem 13 follows that to solve the center problem for the polynomial planar vector field it is necessary to solve the infinity number of first order partial differential equations (29). This problem can be solve in some particular case of the Poincaré– Liapunov integrable differential system.

Proposition 14. *Differential system (28) is Hamiltonian if and only if*

$$(30) \quad \sum_{j=2}^{n+1} \{\Psi_j, g_{n+1-j}\} = 0.$$

Condition (30) is called *divergence condition*.

Proof. System (28) is Hamiltonian if

$$\sum_{j=2}^{n+1} g_{n+1-j} \{\Psi_j, x\} = -\frac{\partial H}{\partial y}, \quad \sum_{j=2}^{n+1} g_{n+1-j} \{\Psi_j, y\} = \frac{\partial H}{\partial x}$$

$H = H(x, y)$ is C^r function with $r \geq 1$. From the compatibility conditions follows that

$$\sum_{j=2}^{n+1} \left(\frac{\partial g_{n+1-j}}{\partial x} \{\Psi_j, x\} + \frac{\partial g_{n+1-j}}{\partial y} \{\Psi_j, y\} \right) = \sum_{j=2}^{n+1} \{\Psi_j, g_{n+1-j}\} = 0.$$

(30). In this case the first integral is

$$\begin{aligned} H(x, y) &= \sum_{j=2}^{m+1} \left(\int_{x_0}^x g_{j+1-j} \{\Psi_j, y\} dx - \int_{y_0}^y g_{j+1-j} \{\Psi_j, x\} |_{x=x_0} dy \right) \\ &:= H_2 + H_3 + \dots + H_{m+1} + \Omega(x, y) = C, \end{aligned}$$

where $H_j = 0$ for $j > n$. \square

Now we shall study the particular case of (28) when $H_3 = H_4 = \dots = H_m = 0$ and $g_1 = g_2 = g_{n-2} = 0$, i.e. the differential system

$$(31) \quad \dot{x} = -\frac{\partial H_{m+1}}{\partial y} - y g_{m-1}, \quad \dot{y} = \frac{\partial H_{m+1}}{\partial x} + x g_{m-1},$$

with the complementary conditions

$$(32) \quad g_{m-1}\{H_2, H_{n+2-m}\} + \{H_2, H_{n+1}\} + \{H_{m+1}, H_{n+2-m}\} = 0,$$

for $n = m, m+1, \dots$, where $g_{m-1} = g_{m-1}(x, y)$ is an arbitrary homogeneous polynomial of degree $m-1$.

Corollary 15. *Differential system (31) is Hamiltonian if and only if*

$$(33) \quad g_{m-1} = \begin{cases} \nu H_2^k, & \text{if } m = 2k+1, \\ 0, & \text{if } m = 2k, \end{cases},$$

where ν is an arbitrary constant.

Proof. Clearly that (31) is Hamiltonian if $\{H_2, g_{m-1}\} = 0$, consequently (33) holds. The Hamiltonian is $H = H_2 + H_3 + \dots + H_{2k} = C$ if $m = 2k$ and $H = H_2 + H_3 + \dots + H_{2k+1} + \nu H_2^{k+1} = C$ if $m = 2k+1$. Conditions (32) in this case hold identically in view of that $H_j = 0$ for $j > 2k$, $m = 2k$ and $H_{2k+2} = \nu H_2^{k+1}$, $H_j = 0$ for $j > 2k+2$, $m = 2k+1$. \square

From conditions (29) and by considering that

$$\int_0^{2\pi} \{H_2, G\}|_{x=\cos t, y=\sin t} dt = \int_0^{2\pi} \frac{dG}{dt} \Big|_{x=\cos t, y=\sin t} dt = 0,$$

for arbitrary C^1 function $G = G(x, y)$, we deduce the relations

$$\int_0^{2\pi} \left(\sum_{j=2}^m g_{j-1} \{H_2, H_{n+2-j}\} + \{H_{m+1}, H_{n+2-m}\} \right) \Big|_{x=\cos t, y=\sin t} dt = 0,$$

for $n = m, m+1, \dots$

From the previous proposition we have the following results.

Corollary 16. *If the polynomial planar vector field $Y = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}$ of degree n has a non-degenerate center at the origin then can be written as*

$$\sum_{j=2}^{m+1} g_{m+1-j} \{\Psi_j, x\} = P(x, y), \quad \sum_{j=2}^{m+1} g_{m+1-j} \{\Psi_j, y\} = Q(x, y).$$

Hence the arbitrary functions g_1, g_2, \dots, g_{m-1} must be satisfies the partial differential equation

$$(34) \quad \sum_{j=2}^m \{\Psi_j, g_{m+1-j}\} = \frac{\partial P(x, y)}{\partial x} + \frac{\partial Q(x, y)}{\partial y}.$$

5. WEAK CONDITIONS OF THE CENTER. GENERALIZATION

Below we need the following concept and result.

Let $\mathbb{R}[x, y]$ be the ring of all real polynomials in the variables x and y , and let

$$\mathcal{X} = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y}$$

be a polynomial vector field of degree m , and let $g = g(x, y) \in \mathbb{R}[x, y]$. Then $g = 0$ is an *invariant algebraic curve* of \mathcal{X} if

$$\mathcal{X}g = P \frac{\partial g}{\partial x} + Q \frac{\partial g}{\partial y} = Kg,$$

where $K = K(x, y)$ is a polynomial of degree at most $m - 1$, which is called the *cofactor* of $g = 0$. If the polynomial g is irreducible in $\mathbb{R}[x, y]$, then we say that the invariant algebraic curve $g = 0$ is *irreducible* and that its *degree* is the degree of the polynomial g . For more details on the so-called Darboux theory of integrability see for instance the Chapter 8 of [11].

For the polynomial system (21) the divergence condition can be weakened as we give in Proposition 10. By applying this result and in view of Theorem 13 we deduce the proof of the next result.

Proposition 17. *If (23) holds then the system (21) is Poincaré–Liapunov integrable.*

Proof. By considering that (23) holds then the system (21) has a center, consequently can be rewritten as follows

$$\begin{aligned} \dot{x} &= -y + X_m = -\frac{\partial H_2 + H_{m+1}}{\partial y} - y g_{m-1}, \\ \dot{y} &= x + Y_m = \frac{\partial H_2 + H_{m+1}}{\partial x} + x g_{m-1}. \end{aligned}$$

Hence

$$\frac{\partial X_m}{\partial x} + \frac{\partial Y_m}{\partial y} = \{H_2, g_{m-1}\}, \quad xX_m + yY_m = \{H_{m+1}, H_2\},$$

consequently the condition (23) becomes

$$\lambda H_2 \{H_2, g_{m-1}\} = \{H_{m+1}, H_2\} \implies \{H_2, H_{m+1} + \lambda H_2 g_{m-1}\} = 0$$

Thus

$$H_{m+1} = \begin{cases} -\lambda H_2 g_{m-1}, & \text{if } m = 2k, \\ -\lambda H_2 g_{m-1} + \nu H_2^{k+1}, & \text{if } m = 2k + 1. \end{cases}$$

Below we study only the case when $\nu = 0$.

From the equations

$$\begin{aligned} \dot{x} &= -y - \frac{\partial H_{m+1}}{\partial y} - y g_{m-1} \\ &= \lambda H_2 \frac{\partial g_{m-1}}{\partial y} - y (1 + (1 - \lambda) g_{m-1}) \\ \dot{y} &= x + \frac{\partial H_{m+1}}{\partial x} + x g_{m-1} \\ &= -\lambda H_2 \frac{\partial g_{m-1}}{\partial x} + x (1 + (1 - \lambda) g_{m-1}) \end{aligned} \tag{35}$$

follows that

$$\begin{aligned} \dot{H}_2 &= \lambda H_2 \{H_2, g_{m-1}\}, \\ \dot{g}_{m-1} &= \{H_2, g_{m-1}\} (1 + (1 - \lambda) g_{m-1}), \end{aligned}$$

thus that the curve $x^2 + y^2 = 0$ and $1 + (1 - \lambda)g_{m-1} = 0$ are invariant curves of the polynomial vector field if $\lambda \in \mathbb{R} \setminus \{0, 1\}$. The cofactors are $\lambda\{H_2, g_{m-1}\}$ and $\{H_2, g_{m-1}\}$ respectively.

The first integral

$$\begin{aligned} F &= (1 + (1 - \lambda)g_{m-1})^{\lambda/(\lambda-1)} H_2 = H_2 - \lambda H_2 g_{m-1} + \dots \\ &= H_2 + f(x, y) = \text{Const.}, \end{aligned} \tag{36}$$

is the Darboux first integral (see [28]), where

$$\lambda = 2/\mu \in \mathbb{R} \setminus \{0, 1\}$$

and $f = f(x, y)$ is a real analytic functions in an open neighborhood of O whose Taylor expansions at O do not contain constant, linear and quadratic terms.

Clearly that in this case we have that

$$H_3 = H_4 = \dots = H_m = 0, \quad H_{m+1} = -\lambda H_2 g_{m-1}, \dots$$

After some computations follows that these functions satisfy the complementary conditions (32).

If $\lambda = 1/m$ then this first integral is the rational function

$$\tilde{F} = \frac{H_2^{m-1}}{(1 + (1 - 1/m)g_{m-1})}. \tag{37}$$

If $\lambda = 1$ then system (35) takes the form

$$\dot{x} = -y + H_2 \frac{\partial g_{m-1}}{\partial y}, \quad \dot{y} = x - H_2 \frac{\partial g_{m-1}}{\partial x}$$

consequently

$$\dot{H}_2 = H_2 \{H_2, g_{m-1}\}, \quad \dot{g}_{m-1} = \{H_2, g_{m-1}\}.$$

Thus $F = H_2 e^{-g_{m-1}}$ is the first integral, which admits the following Taylor expansion in the neighborhood of the origin

$$F = H_2 e^{-g_{m-1}} = H_2 - H_2 g_{m-1} + H_2 \frac{g_{m-1}^2}{2!} + \dots$$

Hence

$$H_3 = H_4 = \dots = H_m = 0, \quad H_{m+1} = -H_2 g_{m-1}, \dots$$

These functions are solutions of the system (32).

The second condition of the center in this case takes the form

$$\begin{aligned} \int_0^{2\pi} (xX_m + yY_m) |_{x=\cos t, y=\sin t} dt &= \int_0^{2\pi} \{H_2, H_{m+1}\} |_{x=\cos t, y=\sin t} dt \\ &= \int_0^{2\pi} \frac{dH_{m+1}}{dt} |_{x=\cos t, y=\sin t} dt = 0. \end{aligned}$$

In short the proposition is proved \square

Corollary 18. *For the system (35) admits an isochronous center if $\lambda = 1/m$ and uniformly isochronous center at the origin if $\lambda = 2/(m+1)$.*

Proof. From (35) follows that

$$x\dot{y} - y\dot{x} = 2H_2 \left(1 + \frac{g_{m-1}}{2} (2 - (m+1)\lambda) \right),$$

which in polar coordinates $x = r \cos \theta$, $y = r \sin \theta$ becomes

$$\dot{\theta} = 1 + \frac{(2 - (m+1)\lambda)}{2} r^{m-1} \Psi(\vartheta),$$

where $\Psi(\vartheta)$ is a 2π periodic function. Hence if $\lambda = 2/(m+1)$ then $\dot{\theta} = 1$. The first integral in this case is a rational function

$$F = \frac{H_2^{m-1}}{\left(1 + \frac{m-1}{m+1} g_{m-1} \right)^2},$$

Consequently the origin is an uniformly isochronous center.

We observe that if $\lambda = 2/(m+1)$ then differential system (35) becomes

$$\begin{aligned}
\dot{x} &= -y + \frac{1}{m+1} \left((x^2 + y^2) \frac{\partial g_{m-1}}{\partial y} - y(m-1)g \right) \\
&= -y + \frac{1}{m+1} \left((x^2 + y^2) \frac{\partial g_{m-1}}{\partial y} - y \left(x \frac{\partial g_{m-1}}{\partial x} + y \frac{\partial g_{m-1}}{\partial y} \right) \right) \\
&= -y + \frac{x}{m+1} \left(x \frac{\partial g_{m-1}}{\partial y} - y \frac{\partial g_{m-1}}{\partial x} \right) := -y + \frac{x}{m+1} \{H_2, g_{m-1}\}, \\
\dot{y} &= x + \frac{y}{m+1} \{H_2, g_{m-1}\}.
\end{aligned}$$

Here we consider that the function g_{m-1} is homogenous function of degree $m-1$.

Now we study the case when $\lambda = 1/m$. From (37) and from the equation $F = C$, where C is an arbitrary constant, follows that

$$r^{m-1} = \frac{C(m-1)}{2m} \Psi(\vartheta) \pm \sqrt{\left(C + \left(\frac{C(m-1)}{2m} \Psi(\vartheta) \right)^2 \right)} := \Phi(\vartheta),$$

thus in view of the periodicity of $\Psi(\vartheta)$ we get

$$\int_0^{2\pi} \frac{d\vartheta}{1 + \frac{(m-1)}{2m} \Phi(\vartheta) \Psi(\vartheta)} = 2\pi.$$

Hence the origin is an isochronous center.

After some computations we can show that differential equations (35) for $\lambda = 1/m$ becomes

$$(38) \quad \dot{x} = -\nu y + \sigma x, \quad \dot{y} = \nu x + \sigma y,$$

where

$$\nu = 1 + \frac{m-1}{2m} g_{m-1}, \quad \sigma = \frac{1}{2m} \{H_2, g_{m-1}\}.$$

Equation (38) in polar coordinates becomes

$$\dot{r} = \frac{r}{2m} \frac{\partial g_{m-1}}{\partial \vartheta}, \quad \dot{\vartheta} = 1 + \frac{m-1}{2m} g_{m-1}.$$

□

Below we shall need the following result.

Proposition 19. *Let $\mathcal{X} = (P, Q)$ be the polynomial vector field where $P = P(x, y)$, and $Q = Q(x, y)$ be the polynomials such that $\max(\deg P, \deg Q) = m$ and such that*

$$(39) \quad \int_0^{2\pi} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) |_{x=\cos t, y=\sin t} dt = 0.$$

Then there exist an unique polynomials $\tilde{H} = \tilde{H}(x, y)$ and $\tilde{g} = \tilde{g}(x, y)$ of degree $m+1$ and $m-1$ respectively for which the following relations hold

$$(40) \quad -\frac{\partial \tilde{H}}{\partial y} - y\tilde{g} = P, \quad \frac{\partial \tilde{H}}{\partial x} + x\tilde{g} = Q.$$

Proof. The polynomials P, Q, \tilde{H} and \tilde{g} can be represented as summa of the homogenous polynomials, ie.

$$\begin{aligned} P &= \sum_{j=1}^m P_j, & Q &= \sum_{j=1}^m Q_j, \\ \tilde{H} &= \sum_{j=1}^{m+1} \tilde{H}_j, & \tilde{g} &= \sum_{j=1}^{m-1} \tilde{g}_j, \end{aligned}$$

From (40) follows that

$$(41) \quad \{H_2, \tilde{g}\} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y},$$

hence

$$\{H_2, \tilde{g}_j\} = \frac{\partial P_j}{\partial x} + \frac{\partial Q_j}{\partial y}.$$

Thus in view of Theorem 4 and condition (39) we deduce that there exist an unique homogenous polynomial $\tilde{g} = \sum_{j=1}^{m-1} \tilde{g}_j$ such that (40) holds.

The function \tilde{H} we determine from the equations

$$\frac{\partial \tilde{H}}{\partial y} = -y\tilde{g} - P, \quad \frac{\partial \tilde{H}}{\partial x} = -x\tilde{g} + Q,$$

where \tilde{g} is a solution of the equation (41). □

Proposition 10 can be generalized as follows.

Proposition 20. (*Generalized weak condition of the center*) Let X and Y be the polynomials of degree at most m i.e.

$$X = \sum_{j=1}^m X_j, \quad Y = \sum_{j=1}^m Y_j,$$

then the origin is a center of vector field (26) if there exist $\mu \in \mathbb{R}$ such that

$$(42) \quad (x^2 + y^2) \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) = \mu (xX + yY),$$

and

$$\int_0^{2\pi} \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) |_{x=\cos t, y=\sin t} dt = 0$$

Proof. Under the given conditions and in view of Proposition 19 there exist the polynomial functions $H = H(x, y)$ and $g = g(x, y)$ of degree $m + 1$ and $m - 1$ respectively such that the following relations hold

$$(43) \quad \begin{aligned} \dot{x} &= -y + X = \{H, x\} + g\{H_2, x\}, \\ \dot{y} &= x + Y = \{H, y\} + g\{H_2, y\}, \end{aligned}$$

Hence

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = \{H_2, g\}, \quad xX + yY = \{H, H_2\},$$

consequently the condition (42) becomes

$$\lambda H_2 \{H_2, g\} = \{H, H_2\} \implies \{H_2, H + \lambda H_2 g\} = 0$$

Thus

$$H = \begin{cases} -\lambda H_2 g, & \text{if } m = 2k, \\ -\lambda H_2 g + \nu H_2^{k+1}, & \text{if } m = 2k + 1. \end{cases}$$

We shall study the case when $\nu = 0$. Analogously to Proposition 17 we prove that the vector field \mathcal{X} is Liapunov-Poincaré integrable with the first integral (Darboux first integral)

$$F = (1 + (1 - \lambda)g)H^{(\lambda-1)/\lambda} \quad \text{if } \lambda \in \mathbb{R} \setminus \{0, 1\},$$

and

$$F = H_2 e^{-\tilde{g}} \quad \text{if } \lambda = 1,$$

where $\tilde{g} = g|_{\lambda=1}$.

Clearly that if the function F is first integral then the function $F^{\lambda/(\lambda-1)}$ is a first integral. By considering that the Taylor expansion of this function at the origin $F^{\lambda/(\lambda-1)}$ and $H_2 e^{-g}$ we obtain

$$H_2 + h.o.t.,$$

thus the origin is a center. \square

Clearly that differential system (43) with $H = -\lambda H_2 g$ becomes

$$\begin{aligned}
 \dot{x} &= -y - \frac{\partial H}{\partial y} - yg = -y + \lambda H_2 \frac{\partial g}{\partial y} - y((1 - \lambda)g) \\
 &= -y + \lambda H_2 \frac{\partial \Psi}{\partial y} - y((1 - \lambda)\Psi) \\
 &\quad \lambda H_2 \frac{\partial g_{m-1}}{\partial y} - y((1 - \lambda)g_{m-1}), \\
 \dot{y} &= x + \frac{\partial H}{\partial x} + xg = x - \lambda H_2 \frac{\partial g}{\partial x} + x(1 - \lambda)g \\
 &= x + \lambda H_2 \frac{\partial \Psi}{\partial x} + x((1 - \lambda)\Psi) \\
 &\quad + \lambda H_2 \frac{\partial g_{m-1}}{\partial x} + x((1 - \lambda)g_{m-1}),
 \end{aligned} \tag{44}$$

where $\Psi = \sum_{j=1}^{m-2} g_j$, with g_j is homogenous function of degree j for $j = 1, \dots, m-2$.

We consider the system

$$\begin{aligned}
 \dot{x} &= -y + \sum_{j=1}^{m-1} X_j + xR_{m-1}, \\
 \dot{y} &= x + \sum_{j=1}^{m-1} Y_j + yR_{m-1},
 \end{aligned} \tag{45}$$

where $R_{m-1} = R_{m-1}(x, y)$ is a convenient nonzero homogenous polynomial of degree $m-1$. Such system are polynomial system with degenerate infinity. This name is due to the fact that in Poincaré compactification of (45) the line at infinity is filled with critical points (see for instance [36])

Proposition 21. *Differential system (44) is a polynomial differential system with degenerate infinity if $\lambda = 2/(m+1)$.*

Proof. From the equation

$$\begin{aligned}
& \lambda H_2 \frac{\partial g_{m-1}}{\partial y} - y((1-\lambda)g_{m-1}) \Big|_{\lambda=2/(m+1)}, \\
&= \frac{1}{m+1} \left((x^2 + y^2) \frac{\partial g_{m-1}}{\partial y} - y(m-1)g_{m-1} \right) \\
&= \frac{1}{m+1} \left((x^2 + y^2) \frac{\partial g_{m-1}}{\partial y} - y \left(x \frac{\partial g_{m-1}}{\partial x} + y \frac{\partial g_{m-1}}{\partial y} \right) \right) \\
&= \frac{x}{m+1} \left(x \frac{\partial g_{m-1}}{\partial y} - y \frac{\partial g_{m-1}}{\partial x} \right) := \frac{x}{m+1} \{H_2, g_{m-1}\},
\end{aligned}$$

Here we consider that the function g_{m-1} is homogenous function of degree $m-1$. Hence (44) becomes

$$\begin{aligned}
(46) \quad \dot{x} &= -y + \frac{2}{m+1} H_2 \frac{\partial \Psi}{\partial y} - y \left(\frac{m-1}{m+1} \Psi \right) \\
&\quad \frac{x}{m+1} \{H_2, g_{m-1}\} := P_{m-1} + \frac{x}{m+1} \{H_2, g_{m-1}\}, \\
\dot{y} &= x - \frac{2}{m+1} H_2 \frac{\partial \Psi}{\partial x} + x \left(\frac{m-1}{m+1} \Psi \right) \\
&\quad + \frac{y}{m+1} \{H_2, g_{m-1}\} := Q_{m-1} + \frac{y}{m+1} \{H_2, g_{m-1}\},
\end{aligned}$$

where $P_{m-1} = P_{m-1}(x, y)$ and $Q_{m-1} = Q_{m-1}(x, y)$ are polynomials of degree $m-1$. Clearly that $\{H_2, g_{m-1}\}$ is a polynomial of degree $m-1$.

The first integral (Darboux first integral) in this case is

$$F = \frac{H_2^{m-1}}{\left(1 + \frac{m-1}{m+1}g\right)^2}$$

□

Now we shall study the case when the center is holomorphic center.

Proposition 22. *The origin is a holomorphic center of the polynomial differential system (43) if and only if*

$$\begin{aligned}
(47) \quad \Delta H + 2g &= 2\Phi, \\
(x^2 + y^2)g + 2mH &= -x \frac{\partial H}{\partial x} - y \frac{\partial H}{\partial y} + (m+1)H \\
&\quad + \int \Phi(x^2 + y^2) d(x^2 + y^2).
\end{aligned}$$

where $\Delta = \frac{\partial^2}{\partial x \partial x} + \frac{\partial^2}{\partial y \partial y}$ and Φ is a function such that

$$\Phi(x^2 + y^2) = \begin{cases} 0, & \text{if } m = 2k, \\ \lambda (x^2 + y^2)^k, & \text{if } m = 2k + 1. \end{cases}$$

Proof. From (25) and (43) we obtain

$$\begin{aligned} y \frac{\partial g}{\partial x} + x \frac{\partial g}{\partial y} &= -2 \frac{\partial^2 H}{\partial y \partial x}, \\ y \frac{\partial g}{\partial y} - x \frac{\partial g}{\partial x} &= \frac{\partial^2 H}{\partial x \partial x} - \frac{\partial^2 H}{\partial y \partial y}. \end{aligned}$$

Hence

$$\begin{aligned} (x^2 + y^2) \frac{\partial g}{\partial x} &= -2y \frac{\partial^2 H}{\partial x \partial y} - x \frac{\partial^2 H}{\partial x \partial x} + x \frac{\partial^2 H}{\partial y \partial y}, \\ (x^2 + y^2) \frac{\partial g}{\partial y} &= -2x \frac{\partial^2 H}{\partial x \partial y} + y \frac{\partial^2 H}{\partial x \partial x} - y \frac{\partial^2 H}{\partial y \partial y}. \end{aligned}$$

Consequently

$$\begin{aligned} (x^2 + y^2) \frac{\partial g}{\partial x} &= x \Delta H - 2 \frac{\partial \Psi}{\partial x}, \\ (x^2 + y^2) \frac{\partial g}{\partial y} &= y \Delta H - 2 \frac{\partial \Psi}{\partial y}, \end{aligned}$$

where $\Psi = x \frac{\partial H}{\partial x} + y \frac{\partial H}{\partial y} - H$, or equivalently

$$\begin{aligned} \frac{\partial}{\partial x} ((x^2 + y^2)g + 2\Psi) &= x (\Delta H + 2g), \\ \frac{\partial}{\partial y} ((x^2 + y^2)g + 2\Psi) &= y (\Delta H + 2g). \end{aligned}$$

Thus

$$\Delta H + g = 2\Phi(x^2 + y^2), \quad (x^2 + y^2)g + 2\Psi = \int \Phi(x^2 + y^2) d(x^2 + y^2).$$

□

Corollary 23. *Is the center is holomorphic center with H homogenous polynomial of degree $m + 1$, then the first integral is*

$$(48) \quad F = \left(1 + \frac{1-m}{m} \Delta H\right) H_2^{1-m}.$$

Proof. If $H = H_{m+1}$ is a homogenous polynomial of degree $m + 1$, then in view of the relation

$$\frac{\partial H_{m+1}}{\partial x} + y \frac{\partial H_{m+1}}{\partial y} = (m + 1) H_{m+1},$$

then by choosing $\Phi = 0$ then from (47) follows

$$\Delta H + 2g = 0, \quad H_2g + mH = 0,$$

thus from Proposition 17 we obtain the existence of the first integral (48). \square

6. ISOCHRONOUS CENTER FOR POLYNOMIAL VECTOR FIELDS WITH DEGENERATE INFINITY

In this section we study the existence of isochronous center for (45) (see Proposition 21.).

Proposition 24. *Quadratic system with degenerate infinity*

$$(49) \quad \dot{x} = -y + x(ax + by), \quad \dot{y} = x + y(ax + by),$$

has at the origin an uniformly isochronous center.

Proof. Indeed, after some computation it is easy to show that the given system admits the invariant algebraic curves

$$g_1 = 1 + ay - bx = 0, \quad g_2 = H_2 = 0, \quad ,$$

with cofactor $K_1 = ax + by$ and $K_2 = 2(ax + by)$ respectively. Consequently the Darboux first integral is

$$F = \frac{\frac{x^2 + y^2}{2}}{(1 + ay - bx)^2},$$

by considering that the following expansion hold $F = \frac{x^2 + y^2}{2} + h.o.t.$ we obtain that the origin is a center and by considering that $x\dot{y} - y\dot{x} = x^2 + y^2$ we easily obtain the proof.

We observe that in view of the relation $\{g_1, g_2\} = ax + by$, system (49) can be rewritten as follows

$$\dot{x} = -y + x\{H_2, g_1\}, \quad \dot{y} = x + y\{H_2, g_1\}.$$

\square

Proposition 25. *cubic system with degenerate infinity*

$$(50) \quad \begin{aligned} \dot{x} &= -y + a(x^2 - y^2) + 2bxy + x(Lx^2 + Mxy - Ly^2), \\ \dot{y} &= x - b(x^2 - y^2) + 2axy + y(Lx^2 + Mxy - Ly^2), \end{aligned}$$

has at the origin an isochronous center at the origin.

Proof. Indeed, after some computations we obtain that

$$F = \frac{\frac{x^2 + y^2}{2}}{(1 + 2ay - 2bx + My^2 + 2Lxy + \kappa(x^2 + y^2))},$$

where κ is an arbitrary constant, is a first integral. In the neighborhood of the origin this function has the following expansion

$$F = \frac{x^2 + y^2}{2} + bx^3 - ay^3 + \frac{b}{3}xy^2 - \frac{a}{3}x^2y + \dots,$$

thus in view of Poincaré Theorem we get that the origin is a center.

Now we prove that this center is isochronous center. In polar coordinates $x = r \cos \vartheta$, $y = r \sin \vartheta$ the first integral becomes

$$F = \frac{\frac{r^2}{2}}{(1 + r(2a \sin \vartheta - 2b \cos \vartheta) + r^2(M \sin^2 \vartheta + 2L \sin \vartheta \cos \vartheta + \kappa))} = F(r, \vartheta).$$

By solving the equation $F(r, \vartheta) = C$, where C is an arbitrary constant, with respect to r we get $r = \Phi(\vartheta)$. On the other hand from the equation

$$xy\dot{y} - y\dot{x} = (1 + ay - bx)(x^2 + y^2) \iff \dot{\vartheta} = 1 + r(a \sin \vartheta - b \cos \vartheta) = 1 + \Phi(\vartheta)(a \sin \vartheta - b \cos \vartheta),$$

after the integration and in view of periodicity of $\Phi(\vartheta)$ we obtain

$$\int_0^{2\pi} \frac{d\vartheta}{1 + \Phi(\vartheta)(a \sin \vartheta - b \cos \vartheta)} = 2\pi.$$

Hence the origin is an isochronous center. \square

Corollary 26. *System (50) with $a = b = 0$ has an uniformly isochronous center at the origin.*

Proof. Indeed, if $a = b = 0$ then the system (50) becomes

$$\begin{aligned} \dot{x} &= -y + x(Lx^2 + Mxy - Ly^2), \\ \dot{y} &= x + y(Lx^2 + Mxy - Ly^2), \end{aligned}$$

Thus $xy\dot{y} - y\dot{x} = x^2 + y^2 \iff \dot{\vartheta} = 1$. Hence, by considering that this system admits the first integral

$$F = \frac{\frac{x^2 + y^2}{2}}{(1 + 2Lxy + \kappa(x^2 + y^2))},$$

which in the neighborhood of the origin has the following Taylor expansion

$$F = \frac{x^2 + y^2}{2} + h.o.t.,$$

we deduced that the origin is an uniformly isochronous center. \square

Remark 27. *In the paper [7] the following results is given.*

Theorem 28. *The cubic polynomial planar vector field with degenerate infinity admits an isochronous center at the origin if and only if this system can be brought to one of the following systems*

(a)

$$\begin{aligned}\dot{x} &= -y + \frac{4}{3}x^2 - 2k_1xy - \frac{x}{3}(4k_1x^2 + 3k_1^2xy), \\ \dot{y} &= x + k_1x^2 - \frac{16}{3}xy - k_1x^2 - \frac{y}{3}(4k_1x^2 + 3k_1^2xy)\end{aligned}$$

(b)

$$\begin{aligned}\dot{x} &= -y + \frac{16}{3}x^2 - 2k_1xy - \frac{4}{3}y^2 + \frac{k_1}{3}x(16x^2 - 3k_1xy - 4y^2), \\ \dot{y} &= x + k_1x^2 + \frac{8}{3}xy - k_1x^2 + \frac{k_1}{3}y(16x^2 - 3k_1xy - 4y^2)\end{aligned}$$

(c)

$$\begin{aligned}\dot{x} &= -y + x^2 - y^2 + x(c_1x^2 + 2c_2xy - c_2y^2), \\ \dot{y} &= x + 2xy + y(c_1x^2 + 2c_2xy - c_2y^2)\end{aligned}$$

(d)

$$\begin{aligned}\dot{x} &= -y + a_1x^2 - a_1y^2 + a_2xy + x(2c_2xy + a_1a_2y^2), \\ \dot{y} &= x + 2a_1xy + a_2y^2 + y(2c_2xy + a_1a_2y^2)\end{aligned}$$

The following question arise. It is possible under linear change of coordinates x, y and a scaling of time t to transformed (50) into the one of the four differential system a), b), c) and d)? If the answer is negative then Theorem 28 give only sufficient conditions.

7. QUADRATIC VECTOR FIELD WITH NON-DEGENERATE CENTER

From Poincaré- Liapunov's work it is known that such system with a center are characterized by a finite number of algebraic independent conditions $D_j = 0$, which are polynomials on the coefficients of the system. The importance of this result is more theoretical than practical. In [24] the following problem was stated: " In order to make an effective use of these conclusions we must answer to the question: Given that right hand members of our equations are polynomial of degree m , to determine $N(m)$ such that all the equations $D_j = 0$ for $j > N(m)$ are consequences of such equalities for $j \leq N(m)$. The problem of characterization of $N(m)$ is still unsolved."

For nondegenerate quadratic (see for instance [3, 34]) and cubic system (see for instance [25, 35]) the center-focus problem has been solved in terms of algebraic equalities satisfied by the coefficients

Proposition 29. *For the quadratic system*

$$(51) \quad \begin{aligned} \dot{x} &= -y - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\ \dot{y} &= x + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2, \end{aligned}$$

the origin is a center if and only if one of the following four conditions holds

- i) $\lambda_4 = \lambda_5 = 0$,
- ii) $\lambda_2 = \lambda_5 = 0$,
- iii) $\lambda_3 - \lambda_6 = 0$,
- iv) $\lambda_5 = 0, \quad \lambda_4 + 5(\lambda_3 - \lambda_6) = 0, \quad \lambda_3 \lambda_6 - 2\lambda_6^2 - \lambda_2^2 = 0$

In [34] the following results is proved.

Proposition 30. *The origin is a center of*

$$(52) \quad \dot{x} = y + ax^2 + bxy + cy^2, \quad \dot{y} = -x + kx^2 + lxy + my^2,$$

if and only if one of the following conditions satisfied:

(i)

$$\begin{aligned} (a+c)(b+2m) - (2a+l)(k+m) &= 0, \\ k(a+c)^3 + (l-a)(a+c)^2(k+m) + (m-b)(a+c)(k+m)^2 - c(k+m)^3 &= 0, \end{aligned}$$

(ii)

$$2a + l = 0, \quad b + 2m = 0.$$

(iii)

$$\begin{aligned} 5(a+c) - (2a+l) &= 0, \\ 5(k+m) - (b+2m) &= 0, \\ c^2 + c(a+c) + k^2 + k(k+m) &= 0. \end{aligned}$$

In the following quadratic differential system the previous two results is comparing

Proposition 31. *The origin is a center of*

$$(53) \quad \begin{aligned} \dot{x} &= -y + ax^2 + \frac{a(3\lambda-2)}{\lambda}y^2 + \frac{2\beta(1-\lambda)}{\lambda}xy := -y * X, \\ \dot{y} &= x + \frac{\beta(3\lambda-2)}{\lambda}x^2 + \frac{2a(1-\lambda)}{\lambda}xy + \beta y^2 := x * Y, \end{aligned}$$

for $\lambda \in \mathbb{R} \setminus \{0, 1\}$ and if $\lambda = 1$ then the origin is a center for

$$(54) \quad \begin{aligned} \dot{x} &= -y + a(x^2 + y^2), \\ \dot{y} &= x + \beta(x^2 + y^2), \end{aligned}$$

where κ is an arbitrary constant.

Proof. After some computations it is possible to show that the function

$$F = (1 + \frac{2(1-\lambda)}{\lambda})(ay - \beta x)^{\lambda/(\lambda-1)} H_2,$$

is a first integral of (53) for arbitrary $\lambda \in \mathbb{R} \setminus \{0, 1\}$. This function have the following Taylor expansion in the neighborhood of the origin

$$F = \frac{x^2 + y^2}{2} + \beta x^3 - ay^3 - \frac{a}{3}(x^2 y + xy^2) + h.o.t.,$$

consequently in view of Poincaré Theorem we obtain that the origin is a center.

For $\lambda = 1$ the first integral is $F_1 = H_2 e^{\beta x - ay} = H_2 + h.o.t.$ Thus the origin is a center in this case. \square

Corollary 32. *Quadratic system (53) satisfies conditions (i) for arbitrary $\lambda \in \mathbb{R} \setminus \{0\}$ and satisfies the Bautin conditions only if $\lambda = \frac{1}{2}$.*

Proof. Indeed, by compare (53) with (52) we obtain that

$$\begin{aligned} a &= a, \quad b = \frac{2\beta(1-\lambda)}{\lambda}, \quad c = \frac{a(3\lambda-2)}{\lambda}, \\ l &= \frac{2a(1-\lambda)}{\lambda}, \quad k = \frac{\beta(3\lambda-2)}{\lambda}, \quad m = \beta. \end{aligned}$$

Consequently conditions (i) satisfies identically.

On the other hand if by compare (53) with (51) we obtain that the unique solution is

$$\lambda_6 - \lambda_3 = 0, \quad \lambda_2 = -\beta, \quad \lambda_3 = -a, \quad \lambda_4 = 4a, \quad \lambda_5 = 4\beta,$$

and $\lambda = 1/2$. Differential system (51) in this case becomes

$$(55) \quad \begin{aligned} \dot{x} &= -y + \frac{\lambda_4}{4}(x^2 - y^2) + \frac{\lambda_5}{2}xy, \\ \dot{y} &= x - \frac{\lambda_5}{4}(x^2 - y^2) + \frac{\lambda_4}{2}xy, \end{aligned}$$

which is equivalent

$$\dot{z} = iz + (\lambda_4 - i\lambda_5)\frac{z^2}{4}, \quad z = x + iy$$

Consequently the singular points $(0, 0)$ and $\left(\frac{4\lambda_5}{\lambda_4^2 + \lambda_5^2}, \frac{-4\lambda_4}{\lambda_4^2 + \lambda_5^2}\right)$ are holomorphic center. \square

Corollary 33. *Quadratic system (53) admits an uniformly isochronous center if $\lambda = 2/3$.*

Proof. Follows from Corollary 18 and Proposition 24. \square

Proposition 34. *Quadratic vector field for which the function (27) is the first integral can be rewritten as follows*

$$(56) \quad \dot{x} = \{H_3 + H_2, x\} + g_1\{H_2, x\}, \quad \dot{y} = \{H_3 + H_2, y\} + g_1\{H_2, y\},$$

with the conditions

$$(57) \quad g_1\{H_2, H_m\} + \{H_2, H_{m+1}\} + \{H_3, H_m\} = 0 \quad \text{for } m = 2, 3, 4, \dots$$

Proof. Follows direct from Theorem (13), with $n = 2$. \square

We determine the representation (56) for differential system (51).

Proposition 35. *System (51) can be rewritten as follows*

$$\begin{aligned} \dot{x} &= \{H_3 + H_2, x\} - (\lambda_5 x - \lambda_4 y)y \\ &= \{H_3 + H_2 + \frac{\lambda_4}{3}y^3 + \frac{\lambda_5}{3}x^3, x\} - \lambda_5 xy, \\ \dot{y} &= \{H_3 + H_2, y\} + (\lambda_5 x - \lambda_4 y)x \\ &= \{H_3 + H_2 + \frac{\lambda_4}{3}y^3 + \frac{\lambda_5}{3}x^3, y\} + \lambda_4 yx, \end{aligned}$$

where H_3 is such that

$$H_3 = \frac{1}{3}(\lambda_2 + \lambda_5)x^3 + \lambda_3 x^2 y - \frac{1}{3}(\lambda_4 + \lambda_6)y^3 - \lambda_2 xy^2.$$

Proof. In view of the relation

$$\int_0^{2\pi} \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) \Big|_{x=\cos t, y=\sin t} dt \equiv 0$$

where X and Y are quadratic polynomials given by the formula (53), we deduce that for Bautin quadratic system we have that

$$\begin{aligned} (58) \quad -y - \frac{\partial H_3}{\partial y} - yg_1 &= -y - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\ x + \frac{\partial H_3}{\partial x} + xg_1 &= x + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2, \\ g_1 &= Ax + By. \end{aligned}$$

Hence we obtain

$$\{H_2, g_1\} = \lambda_5 y + \lambda_4 x,$$

thus

$$g_1 = \lambda_4 y - \lambda_5 x.$$

From (58) we have

$$\begin{aligned} \frac{\partial H_3}{\partial y} &= \lambda_3 x^2 - 2\lambda_2 xy - (\lambda_6 + \lambda_4)y^2, \\ \frac{\partial H_3}{\partial x} &= (\lambda_2 - \lambda_5)x^2 + 2\lambda_3 xy - \lambda_2 y^2, \end{aligned}$$

After integration we deduce the expression for H_3 . Thus after some computation we deduce the proof of the proposition. \square

By solving the equation (57) for $m = 3$ i.e.

$$(\lambda_4 y - \lambda_5 x)\{H_2, H_3\} + \{H_2, H_4\} = 0,$$

we deduce that

$$\begin{aligned} H_4 &= \left(\frac{\lambda_2 \lambda_5}{2} - \frac{\lambda_2^2}{4} - \frac{\lambda_4 \lambda_6}{2} + \frac{\lambda_5^2}{4} \right) x^4 + \lambda_4 \lambda_2 x y^3 \\ &\quad + \left(-\frac{\lambda_2 \lambda_5}{2} - \frac{\lambda_2^2}{4} - \frac{3\lambda_4 \lambda_6}{2} + \frac{\lambda_5^2}{4} \right) + \lambda_5 \lambda_6 y x^3 + a_2(x^2 + y^2)^2, \end{aligned}$$

if $\lambda_3 - \lambda_6 = 0$, and $\lambda_5 \neq 0$. If $\lambda_5 = 0$ then

$$\begin{aligned} H_4 &= \left(-\frac{\lambda_4^2}{4} - \frac{\lambda_4 \lambda_6}{4} - \frac{\lambda_4 \lambda_3}{4} \right) x^4 + \lambda_4 \lambda_2 x y^3 + \\ &\quad + \left(-\frac{\lambda_4^2}{2} - \frac{\lambda_4 \lambda_6}{2} - \lambda_4 \lambda_3 \right) x^2 y^2 + a_2(x^2 + y^2)^2, \end{aligned}$$

Now we deduce Poincaré–Liapunov integrable quadratic systems.

Corollary 36. *Quadratic system (51) is Hamiltonian if and only if*

$$\lambda_4 = \lambda_5 = 0, \quad H_j = 0 \quad \text{for } j > 3.$$

Proof. Indeed, from (30) for $n = 2$ follows that $\{H_2, g_1\} = 0$, consequently $g_1 = g_1(r)$. By considering that g_1 is a homogenous polynomial of degree one we deduce that $g_1 = 0$. Consequently the first integral is $V(x, y) = H_2 + H_3$ consequently $H_j = 0$ for $j > 3$. In short the corollary is proved. \square

Corollary 37. *(Weak condition of the center for quadratic system)*

System (51) is Poincaré–Liapunov integrable if one of the following two condition holds.

i)

$$\lambda = \frac{1}{2}, \quad \lambda_3 - \lambda_6 = 0, \quad \lambda_3 = -\frac{\lambda_4}{4} \quad \lambda_2 = -\frac{\lambda_5}{4},$$

and

ii)

$$\lambda_2 = \lambda_5 = 0, \quad \lambda_3 = -\frac{\lambda\lambda_4}{2}, \quad \lambda_6 = \frac{3\lambda\lambda_4}{2} - \lambda_4,$$

for $\lambda \in \mathbb{R} \setminus \{0, 1/2\}$.*Proof.* Indeed, from the equation

$$\begin{aligned} 0 &= H_3 + \lambda(\lambda_5 x - \lambda_4 y)H_2 = \frac{1}{3}(-\lambda_2 + \lambda_5)x^3 + \lambda_3 x^2 y + \frac{1}{3}(\lambda_4 + \lambda_6)y^3 - \lambda_2 xy^2 \\ &\quad + \frac{\lambda}{2}(\lambda_5 x - \lambda_4 y)(x^2 + y^2) \\ &= \left(\frac{2\lambda_2}{3} + \frac{\lambda_5}{3} - \frac{\lambda\lambda_5}{2}\right)x^3 + \left(\lambda_3 + \frac{\lambda\lambda_4}{2}\right)yx^2 - \left(\frac{\lambda\lambda_5}{2} + \lambda_2\right)xy^2 \\ &\quad + \left(\frac{\lambda\lambda_4}{2} - \frac{\lambda_4}{3} - \frac{\lambda_6}{3}\right)y^3 = \sum_{j+k=3} b_{jk}x^j y^k, \end{aligned}$$

and by requiring that $b_{jk} = 0$ for $j, k = 1, 2, 3$ we obtain the proof.If $\lambda \in \mathbb{R} \setminus \{0, 1/2, 1\}$ then the first integral is

$$F = (1 + (\lambda - 1)\lambda_4 y)H_2^{(\lambda-1)/\lambda}.$$

If $\lambda = 1$ then the first integral is

$$F = H_2 e^{-\lambda_4 y}.$$

If $\lambda = 1/2$ the first integral is

$$F = \frac{1 + 1/2(\lambda_5 x - \lambda_4 y)}{H_2}.$$

After some computations we can show that the quadratic system for the case when $\lambda = 1/2$ can be written as (55). Consequently the origin is isochronous center. \square

We observe that if we compute the condition of the existence the holomorphic center for $m = 2$ i.e. $\Delta H_3 + 2g_1 = 0$ and $(x^2 + y^2)g_1 + 4H_3 =$

0 we obtain

$$\begin{aligned}\Delta H_3 + 2g_1 &= 2(\lambda_3 - \lambda_6)y = 0, \\ (x^2 + y^2)g_1 + 4H_3 &= \frac{1}{\lambda_3}(\lambda_5 + 4\lambda_2)x^3 + (4\lambda_3 + \lambda_4)yx^2 - (\lambda_5 + 4\lambda_2)xy^2 \\ &\quad - \frac{1}{3}(\lambda_4 + 4\lambda_6)y^3 = 0.\end{aligned}$$

Clearly these conditions hold if and only if

$$\lambda_3 - \lambda_6 = 0, \quad \lambda_2 = -\frac{\lambda_5}{4}, \quad \lambda_4 = -\frac{\lambda_4}{4},$$

under these conditions system (51) coincide with system 55.

Now we study the representation (56) for the quadratic system (52). After some computations we can prove that the function H_3 and g_1 are

$$\begin{aligned}H_3 &= -\frac{2m+b+k}{3}x^3 - mxy^2 + ax^2y - \frac{2a+l+c}{3}y^3, \\ g_1 &= (2m+b)x - (2a+c)y.\end{aligned}$$

The weak conditions of the center for quadratic system (52) produce the quadratic system (53). if $\lambda \in \mathbb{R} \setminus \{0, 1\}$ and (54) if $\lambda = 1$. Clearly, if $g_1 \equiv 0$, i.e. $2m+b=0$ and $c+2a=0$ then the system is Hamiltonian.

Now we study the reversible quadratic system.

Proposition 38. *The most general reversible quadratic system with non-degenerated center, invariant under the transformation $(x, -y - t) \longrightarrow (x, y, t)$ is*

$$\dot{z} = i(z + b_{20}z^2 + b_{02}\bar{z}^2 + b_{11}z\bar{z}),$$

or equivalently

$$(59) \quad \dot{x} = -y - 2\alpha xy, \quad \dot{y} = x + ry^2 + sx^2,$$

where $\alpha = b_{02} - b_{20}$, $s = b_{11} + b_{02} + b_{20}$, $r = b_{11} - b_{02} - b_{20}$.

Proof. Follows from (18) with $m = 2$. The following relations holds

$$b_{11} = (s + r)/2, \quad b_{02} = \frac{\alpha}{2} + \frac{s}{4} - \frac{r}{4}, \quad b_{20} = \frac{s}{4} - \frac{\alpha}{2} - \frac{r}{4}.$$

□

Proposition 39. *Quadratic differential system (59) is Poincaré-Liapunov integrable, with the first integral*

$$\begin{aligned}F &= (y^2 + \mu_1(4rs\alpha^3(1 + 2\alpha s)x^2 + 4\alpha^2s(2\alpha^2s + 2\alpha - r)x - 2\alpha^2s - 2\alpha + r)) \cdot \\ &\quad (1 + 2\alpha x)^{2\alpha s} = \text{Const.},\end{aligned}$$

if $r\alpha(1+\alpha s)(1+2\alpha s) \neq 0$, where $\mu_1 = \frac{1}{16r\alpha^4(1+\alpha s)(1+2\alpha s)}$, and

$$F = (y^2 + \mu_2(4r\alpha^3(1+2\alpha r)x^2 - 4r\alpha^2(1+2\alpha^2+2r\alpha^3)x + 1 + 2r\alpha^3 + 2\alpha^2)) \cdot (1+2\alpha x)^{2\alpha r} = \text{Const.},$$

if $1+\alpha s = 0$ and $r\alpha(1+\alpha r)(1+2\alpha r) \neq 0$, where $\mu_2 = \frac{1}{16r\alpha^4(1+\alpha r)(1+2\alpha r)}$. Analogously we can study the case when $1+2\alpha s = 0$.

Proof. After the integration equation (59) we deduce the existence of the given first integral. By considering that the Taylor development in the neighborhood of the origin is

$$F = \frac{1}{2}(x^2 + y^2) + h.o.t.,$$

thus we obtain that the given quadratic system is Poincaré–Liapunov integrable. \square

Proposition 40. *The most general quadratic system with non-degenerated center, invariant under the transformation $(-x, y, -t) \rightarrow (x, y, t)$ is*

$$\dot{z} = iz + \beta_{20}z^2 + \beta_{11}z\bar{z} + \beta_{02}\bar{z}^2,$$

where β_{jk} are real constants, or equivalently

$$(60) \quad \dot{x} = -y + bx^2 + cy^2, \quad \dot{y} = x + \beta xy,$$

where

$$\beta = 2(\beta_{20} - \beta_{02}), \quad b = \beta_{20} + \beta_{02} + \beta_{11}, \quad c = -\beta_{20} - \beta_{02} + \beta_{11},$$

Proof. Follows from Proposition 7 with $m=2$ and $a_{00} = a_{01} = 0$. \square

Proposition 41. *Differential system (60) is Poincaré–Liapunov integrable.*

Proof. Indeed, if $b(b-2\beta)(b-\beta) \neq 0$ and $\beta > 0$ then system (60) admits the first integral

$$F = \frac{x^2 + \lambda(bc(b-\beta)y^2 + b(2\beta+2c-b)y + 2\beta+2c-b)}{(1+\beta y)^{b/\beta+1}},$$

where $\lambda = \frac{1}{b(b-2\beta)(b-\beta)}$. By developing in Taylor series we obtain the following development

$$F = 2x^2 + y^2 - 2(b+c+\beta)y^3 + 3(b+\beta)(b+2\beta+2c)y^5 + \dots,$$

thus the given quadratic is Poincaré–Liapunov integrable.

If $b = 2\beta$ then (60) admits the first integral

$$F = \frac{x^2}{1 + \beta y} + \frac{c + \beta}{2\beta^3(1 + \beta y)^2} + \frac{2c + \beta}{\beta^3(1 + \beta y)} + \frac{c}{\beta^2} \log(1 + \beta y).$$

The Taylor expansion in the neighborhood of the origin is

$$F = 2x^2 + y^2 - (5\beta + c)y^3 + 18\beta(2\beta + c)y^4 - 48\beta^2(5\beta + 3c)y^5 + \dots,$$

thus the given quadratic system is Poincaré–Liapunov integrable for $b = 2\beta$.

The case when $b(b - \beta) = 0$, can be studied analogously. Thus the proposition is proved. \square

It is well-known the following results (see for instance [18, 27]).

Proposition 42. *The quadratic vector field*

$$\dot{x} = y + X_2(x, y), \quad \dot{y} = -x + Y_2(x, y),$$

has an isochronous center at the origin if and only if

$$\begin{aligned} (61) \quad & i) \quad \dot{x} = y(1 + x), \quad \dot{y} = -x + y^2, \\ & ii) \quad \dot{x} = y(1 + x), \quad \dot{y} = -x + \frac{y^2}{2} - \frac{x^2}{2} \iff \\ & \quad \dot{z} = i(z - \frac{z^2}{2}), \\ & iii) \quad \dot{x} = y(1 + x), \quad \dot{y} = -x + \frac{y^2}{4}, \\ & iv) \quad \dot{x} = y(1 + x), \quad \dot{y} = -x + 2y^2 - \frac{x^2}{2}, \end{aligned}$$

In [21] the author obtained a different identification of the quadratic system with an isochronous center at the origin, based directly on the coefficients, namely (see for instance [35])

Proposition 43. *The quadratic differential system*

$$(62) \quad \dot{x} = -y + ax^2 + bxy + cy^2, \quad \dot{y} = x + kx^2 + lxy + my^2,$$

has an isochronous center at the origin if and only if one of the following conditions is satisfied

- i) $a - l = 0, \quad c = k = 0, \quad b = m.$*
- ii) $a - c - l = 0, \quad a + c = 0, \quad b + k - m = 0, \quad k + m = 0.$*

iii)

$$\begin{aligned} 4a + 5c - l &= 0, & 3b - 6k - 4m &= 0, \\ \alpha(\alpha^2 + \gamma^2) + \beta(\beta^2 - 3\delta^2) &= 0, \\ \gamma(\alpha^2 + \gamma^2) - 27(3\beta^2 - \delta^2)\delta &= 0, \end{aligned}$$

where $\alpha = b + k - m$, $\beta = -b + 3k + m$, $\gamma = -a + c + l$, $\delta = -a - 3c + l$.

By solving relations i) and ii) we obtain the following quadratic vector field

$$\dot{x} = -y + x(lx + my), \quad \dot{y} = x + y(lx + my),$$

and

$$\dot{x} = -y + c(x^2 - y^2) + bxy, \quad \dot{y} = x + \frac{b}{2}(y^2 - x^2) - 2cxy,$$

which is equivalent

$$\dot{z} = iz - (c + i\frac{b}{2})z^2, \quad z = x + iy.$$

For the conditions iii) after some computations we obtain the following quadratic system

$$\begin{aligned} \dot{x} &= -y + \frac{3\delta - \gamma}{6}x^2 + \frac{3\alpha - 5\beta}{2}xy + \frac{\gamma - \delta}{4}y^2, \\ \dot{y} &= x + \frac{\alpha + \beta}{4}x^2 + \frac{7\gamma + 9\delta}{12}xy + \frac{\alpha - 3\beta}{4}y^2, \end{aligned}$$

where α, β, γ and δ are constants such that

$$\alpha(\alpha^2 + \gamma^2) + \beta(\beta^2 - 3\delta^2) = 0, \quad \gamma(\alpha^2 + \gamma^2) - 27\delta(3\beta^2 - \delta^2) = 0.$$

Remark 44. From the previous results we deduce the following remarks.

- (a) The representation (56) for the reversible quadratic system we obtain with the functions

$$g_1 = -2(s + \alpha)x, \quad H_3 = \frac{1}{2}(x^2 + y^2) + sxy^2 + \frac{r + 2s + 2\alpha}{3}x^3.$$

The solutions of the equation $H_3 + \lambda g_1 H_2 = 0$ in this case are $s = \frac{1}{3}$ and $\alpha = \frac{\lambda - 1}{3\lambda}$.

- (b) From Proposition 42 follows that any quadratic differential system with isochronous center at the origin under linear change of coordinates x, y and a scaling of time t can be transformed any reversible quadratic system of the type (61) which is a particular case of the system (59).

- (c) *The following question arise, It is possible to show that under linear change of coordinates x, y and a scaling of time t to transform quadratic non-reversible differential system (55) with two isochronous center and with $\lambda_4 \neq 0$ in to the one of the reversible quadratic system (61) ? If the answer is negative then Proposition 42 and 43 are not equivalent.*
- (iii) *Differential system (62) with the conditions (ii) in complex coordinates becomes*

$$\dot{z} = iz + \frac{l - ib}{2}z^2,$$

which coincide with differential system (55) if we choose $\lambda_4 = 2l$ and $\lambda_5 = 2b$.

8. CUBIC VECTOR FIELD WITH NON-DEGENERATE CENTER

Now we apply Theorem 13 to study the cubic planar vector field with first integral (27).

Proposition 45. *Cubic polynomial planar differential system*

$$\dot{x} = -y + X_2 + X_3 = P, \quad \dot{y} = x + Y_2 + Y_3,$$

where $X_j = X_j(x, y)$ and $Y_j = Y_j(x, y)$ are homogenous polynomial of degree j for $j = 2, 3$, has a center in the origin if and only if

$$(63) \quad \begin{aligned} \dot{x} &= \{H_2 + H_3 + H_4, x\} + g_1\{H_2 + H_3, x\} + g_2\{H_2, x\}, \\ \dot{y} &= \{H_2 + H_3 + H_4, y\} + g_1\{H_2 + H_3, y\} + g_2\{H_2, y\}, \end{aligned}$$

where H_j and g_k are homogenous polynomial of degree j and k respectively, where $j = 2, 3, 4$ and $k = 1, 2$, which satisfies the partial differential equations of first degree

$$(64) \quad g_2\{H_2, H_{n-1}\} + g_1\{H_2, H_n\} + \{H_2, H_{n+1}\} + \{H_4, H_{n-1}\} = 0,$$

for $n = 3, 4, \dots$.

Proof. System (28) and (29) for $n = 3$ coincide with (63) and (64). \square

We shall study the case when the cubic differential system is the following

$$(65) \quad \begin{aligned} \dot{x} &= -y + ax^2 + by^2 + cxy + Ax^3 + Bx^2y + Cxy^2 + Dy^3 := -y + X, \\ \dot{y} &= x + \alpha x^2 + \beta y^2 + \gamma xy + Kx^3 + Lx^2y + Mxy^2 + Ny^3 := x + Y. \end{aligned}$$

Corollary 46. *Differential system (65) can be rewritten in the form (63) if and only if*

$$(66) \quad 3(N + A) + L + C = 0$$

Proof. If (63) and (66) hold then the functions g_1 , g_2 , H_3 and H_4 are

$$\begin{aligned}
 H_3 &= \frac{\alpha + 2\beta + c}{3}x^3 - \frac{2a + \gamma + b}{3}y^3 - ax^2y + \beta xy^2, \\
 H_4 &= \frac{B + M + K}{4}x^4 - \frac{D}{4}y^4 - \Lambda(x^2 + y^2)^2, \\
 (67) \quad &+ \frac{M}{2}x^2y^2 - Ax^3y - \frac{(3A + L + C)}{3}y^3x, \\
 g_1 &= -(2\beta + c)x + (2a + \gamma)y, \\
 g_2 &= -(B + M)x^2 + (3A + L)xy + \Lambda(x^2 + y^2),
 \end{aligned}$$

where Λ is an arbitrary constant. □

From (12) we deduce the equation $g_1\{H_2, H_3\} = 0$. By inserting g_1 and H_3 from (67) into this equation we obtain

$$a = 0, \quad \text{and} \quad \gamma\alpha = 0, \quad \gamma(\gamma + b) = 0.$$

Consequently condition (66), functions H_3, H_4 and g_1, g_2 takes the form if $a = 0$ and $\gamma = 0$

$$\begin{aligned}
 0 &= N + A + \frac{L + C}{3}, \quad \beta + b = 0, \\
 H_3 &= \frac{\alpha + c}{3}x^3 - \frac{b}{3}y^3, \\
 H_4 &= \frac{B + D + M + K + c(\alpha + c)}{4}x^4 + \frac{D + M}{2}x^2y^2 + l_{04}(x^2 + y^2)^2 \\
 &\quad - Ax^3y - (A + \frac{L + C}{3})xy^3, \\
 g_1 &= -cx, \\
 g_2 &= -(B + M)x^2 + (3A + L)xy \\
 &\quad + \nu_0(x^2 + y^2),
 \end{aligned}$$

where $\nu_0 = \nu|_{a=0, \gamma=0}$. If $a = 0$, $\alpha = 0$ and $\gamma + b = 0$ then

$$\begin{aligned}
0 &= N + A + \frac{L+C}{3}, \quad \beta + b = 0, \\
H_3 &= 0, \\
H_4 &= \frac{B+D+M+K+c^2}{4}x^4 + \frac{D+M}{2}x^2y^2 - Ax^3y \\
&\quad - (A + \frac{L+C}{3})xy^3 + \Lambda(x^2+y^2)^2 \\
&= \frac{B+D+M+K}{4}x^4 + \frac{D+M}{2}x^2y^2 - Ax^3y + Nxy^3 + \Lambda(x^2+y^2)^2, \\
g_1 &= \gamma y, \\
g_2 &= -(B+M)x^2 + (3A+L)xy \\
&\quad + \nu_1(x^2+y^2),
\end{aligned}$$

where Λ is an arbitrary constant, $\nu_0 = \nu|_{a=0, \alpha=0, \gamma+b=0}$.

Corollary 47. *Cubic system (65) under the condition (66) is Hamiltonian if and only if*

$$c = 2a, \quad \gamma = -2a, \quad B + M = L + 3A = 0, \quad 3N + C = 0.$$

Proof. From Proposition 14 for $n = 3$ we obtain

$$\{H_2 + H_3, g_1\} + \{H_2, g_2\} = 0,$$

holds if

$$c = 2a, \quad \gamma = -2a, \quad B + M = L + 3A = 0,$$

thus in view of the conditions $0 = N + A + \frac{L+C}{3}$, $\beta + b = 0$, we deduce the proof of the corollary. The Hamiltonian is

$$\begin{aligned}
H &= \frac{D+K}{4}x^4 + \frac{D+M}{2}x^2y^2 - Ax^3y - \frac{C}{3}xy^3 \\
&\quad + \Lambda(x^2+y^2)^2 + \frac{\alpha}{3}x^3 - \frac{b}{3}y^3.
\end{aligned}$$

□

In order to illustrate the previous results we study the Kukles' system

$$\begin{aligned}
\dot{x} &= -y, \\
\dot{y} &= x + \alpha x^2 + \beta y^2 + \gamma xy + Kx^3 + Lx^2y + Mxy^2 + Ny^3,
\end{aligned}$$

Example 48. *The Kukles' system can be written in the form (63) if $3N + L = \gamma\alpha = 0$, $\beta = 0$ with*

$$(68) \quad \begin{aligned} H_4 &= \frac{\gamma^2 - M}{2}y^4 + Nxy^2 + (l_{04} + \frac{M - \gamma^2}{4})(x^2 + y^2)^2, \\ H_3 &= \frac{\alpha}{3}x^3 - \frac{\gamma}{3}y^3, \\ g_1 &= \gamma x, \quad g_2 = My^2 + 3Nxy + (l_{04} + \frac{M - \gamma^2}{4})(x^2 + y^2) \end{aligned}$$

Under the given conditions formula (63) holds with the functions H_4, H_3, g_1 and g_2 given in (68).

Proposition 49. *The most general cubic differential system invariant under the change $(x, -y, -t) \longrightarrow (x, y, t)$ is*

$$\dot{z} = i \left(b_{10}z + b_{01}\bar{z} + b_{20}z^2 + b_{02}\bar{z}^2 + b_{11}z\bar{z} + b_{30}z^3 + b_{03}\bar{z}^3 + b_{21}z^2\bar{z} + b_{12}z\bar{z}^2 \right),$$

where $b_{jk} \in \mathbb{R}$, or equivalently

$$\begin{aligned} \dot{x} &= -(b_{10} - b_{01})y - 2(b_{20} - b_{02})yx + (b_{21} + b_{03} - b_{30} + b_{12})y^3 \\ &\quad + (3(b_{30} + b_{03}) + b_{21} - b_{12})yx^2, \\ \dot{y} &= (b_{10} + b_{01})x + (b_{30} + b_{03} + b_{12} + b_{21})x^3 \\ &\quad + (b_{20} + b_{02} + b_{11})x^2 + (b_{11} - b_{20} - b_{02})y^2 \\ &\quad + (b_{21} + b_{12} - 3(b_{30} + b_{03}) + b_{21} - b_{12})y^2x \end{aligned}$$

Proof. Follows direct from Proposition 7 with $m = 3$. \square

Proposition 50. *The most general reversible cubic differential system invariant under the change $(x, -y, -t) \longrightarrow (x, y, t)$ is*

$$\dot{z} = i \left(\alpha_{10}z + \alpha_{01}\bar{z} + \alpha_{30}z^3 + \alpha_{03}\bar{z}^3 + \alpha_{21}z^2\bar{z} + \alpha_{12}z\bar{z}^2 \right) + \beta_{20}z^2 + \beta_{02}\bar{z}^2 + \beta_{11}z\bar{z},$$

where $\alpha_{jk}, \beta_{jk} \in \mathbb{R}$, or equivalently

$$(69) \quad \begin{aligned} \dot{x} &= -(\alpha_{10} - \alpha_{01})y + (\beta_{20} + \beta_{02} + \beta_{11})x^2 + (\beta_{11} + \beta_{02} - \beta_{20})y^2 \\ &\quad + (\alpha_{21} + \alpha_{03} - \alpha_{30} + \alpha_{12})y^3 + (3(\alpha_{30} + \alpha_{03}) + \alpha_{21} - \alpha_{12})yx^2, \\ \dot{y} &= (\alpha_{10} + \alpha_{01})x + (\alpha_{30} + \alpha_{03} + \alpha_{12} + \alpha_{21})x^3 + 2(\beta_{20} - \beta_{02})xy \\ &\quad + (\alpha_{21} + \alpha_{12} - 3(\alpha_{30} + \alpha_{03}) + \alpha_{21} - \alpha_{12})y^2x \end{aligned}$$

Proof. Follows direct from Proposition 7 with $m = 3$. \square

Corollary 51. *Assume that in (69) $\alpha_{10} = 1$ and $\alpha_{01} = 0$ then the origin is a center.*

9. QUASI-HOMOGENOUS CUBIC VECTOR FIELD WITH NON-DEGENERATE CENTER

For non-degenerate quasi-homogenous cubic system (see for instance [25, 35]) the center-focus problem has been solved in terms of algebraic equalities satisfied by the coefficients.

Proposition 52. *For the cubic system*

$$(70) \quad \begin{aligned} \dot{x} &= y + Ax^3 + Bx^2y + Cxy^2 + Dy^3 := -y + X, \\ \dot{y} &= -x + Kx^3 + Lx^2y + Mxy^2 + Ny^3 := x + Y, \end{aligned}$$

the origin is a center if and only if one of the following sets of conditions hold

$$\begin{aligned} i) \quad & 3A + L + C + 3N = 0, \\ & (3A + L)(B + D + K + M) - 2(A - N)(B + M) = 0, \\ & 2(A + N)((3A + L)^2 - (B + M)^2) \\ & + (3A + L)(B + M)(B + K - D - M) = 0, \\ ii) \quad & 3A + L + C + 3N = 0, \\ & 2A + C - L - 2N = 0, \\ & B + 3D - 3K - M = 0, \\ & B + 5D + 5K + M = 0, \\ & (A + 3N)(3A + N) - 16DK = 0 \end{aligned}$$

Now we shall study the quasi-homogenous cubic vector field (70) by applying the results obtained from the solution of the inverse problem of the center problem.

Proposition 53. *Differential system (70) is Poincaré–Liapunov integrable if*

$$(71) \quad \begin{aligned} L + C &= A + N = 0, \quad C = -\frac{N(-2 + 3\lambda)}{\lambda}, \\ D &= \frac{(2\lambda - 1)}{\lambda}B + \frac{(2\lambda - 1)r}{\lambda}, \\ K &= \frac{(2\lambda - 1)r}{\lambda}, \quad M = -\frac{(\lambda - 1)}{\lambda}B + 2\frac{(1 - 2\lambda)r}{\lambda^2}, \end{aligned}$$

where $\lambda \in \mathbb{R} \setminus \{0\}$ and r is an arbitrary constant.

Proof. In view of the relation

$$\int_0^{2\pi} \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) |_{x=\cos t, y=\sin t} dt = L + C + 3(A + N) = 0,$$

where X and Y are polynomials given by the formula (70), and after the integration the equations

$$\begin{aligned} -y - \frac{\partial H_4}{\partial y} - yg_2 &= -y + Ax^3 + Bx^2y + Cxy^2 + Dy^3, \\ x + \frac{\partial H_4}{\partial x} + xg_2 &= x + Kx^3 + Lx^2y + Mxy^2 + Ny^3 \end{aligned}$$

we obtain

$$\begin{aligned} H_4 &= -\frac{K+B+D+M}{4}y^4 - \frac{K+B}{2}x^2y^2 - Ax^3y + Nxy^3 + r(x^2+y^2)^2, \\ g_2 &= (M+B+K)y^2 - (3N+C)xy - (4r-K)(x^2+y^2) \end{aligned}$$

The relation $H_4 = -\lambda g_2 H_2$ holds if (71) and $3(A+N) + L + C = 0$ take place.

Differential system (70) under conditions (71) takes the form

$$\begin{aligned} \dot{x} &= -y - Nx \left(x^2 + \frac{(3\lambda-2)}{\lambda}y^2 \right) + By \left(x^2 + \frac{2\lambda-1}{\lambda}y^2 \right) \\ &\quad + 2r \frac{(2\lambda-1)(\lambda-1)}{\lambda^2} y^3, \\ \dot{y} &= x + Ny \left(y^2 + \frac{3\lambda-2}{\lambda}x^2 \right) - \frac{\lambda-1}{\lambda} Bxy^2 \\ &\quad + 2r \frac{2\lambda-1}{\lambda^2} (\lambda x^2 + y^2), \end{aligned}$$

The function g_2 in this case takes the form

$$g_2 = -\frac{1}{\lambda} \left(2r(y^2)x^2 - (B - \frac{2}{\lambda}r)y^2 - \frac{2N}{\lambda}xy \right).$$

From Corollary 17 we obtain that this system is Poincaré–Liapunov integrable with the first integral

$$F = \left(1 + (1-\lambda) \left(\frac{B}{\lambda}y^2 - \frac{2N}{\lambda}xy - \frac{2r}{\lambda}(x^2 + \frac{\lambda-1}{\lambda}) \right) \right) (x^2+y^2)^{(\lambda-1)/\lambda},$$

if $\lambda \in \mathbb{R} \setminus \{0, 1\}$ and

$$F = H_2 e^{1/2(By^2 - 2Nxy - 2rx^2)},$$

if $\lambda = 1$. □

Corollary 54. *The quasi-homogenous cubic planar vector field with isochronous center is*

$$\begin{aligned}\dot{x} &= -y + Nx(3y^2 - x^2) + By(x^2 - y^2) + \frac{4r}{3}y^3, \\ \dot{y} &= x + Ny(y^2 - 3x^2) - 2Bxy^2 - \frac{2r}{3}x(x^2 + 3y^3).\end{aligned}$$

Proof. Indeed, from Corollary 18 with $m = 3$ and $\lambda = 1/3$ we obtain the proof. \square

Corollary 55. *The quasi-homogenous cubic planar vector field with uniformly isochronous center is*

$$\begin{aligned}\dot{x} &= -y + Nx(x^2 - y^2) + Byx^2, \\ \dot{y} &= x - Ny(x^2 - y^2) + Bxy^2,\end{aligned}$$

Proof. Indeed, from Corollary 18 with $m = 3$ $\lambda = 1/2$ we obtain the proof. \square

Corollary 56. *The quasi-homogenous cubic planar vector field with holomorphic center is*

$$(72) \quad \begin{aligned}\dot{x} &= -y - Nx(x^2 - 3y^2) + 2ry(3x^2 - y^2), \\ \dot{y} &= x + Ny(y^2 - 3x^2) + 2rx(3y^2 - x^2),\end{aligned}$$

or equivalently

$$\dot{z} = iz - (N + 2ir)z^3.$$

and uniformly isochronous center is

$$(73) \quad \begin{aligned}\dot{x} &= -y + x(N(y^2 - x^2) + Bxy), \\ \dot{y} &= x + y(N(y^2 - x^2) + Bxy),\end{aligned}$$

respectively

Proof. By solving the equations $3H_4 + g_2H_2 = 0$ and $\Delta H_4 + 2g_2 = 0$ we after some computations we obtain the proof of the corollary. We observe that the first integral in this case is

$$F = \frac{1 - 4(r(x^2 - y^2) + Nxy)}{H_2^2} = Const..$$

The proof of (73) follows from Corollary 18 with $m = 3$. \square

Now we shall study the reversible cubic system.

Now we shall study the quasi-homogenous reversible cubic differential system

$$(74) \quad \dot{z} = i(b_{10}z + b_{01}\bar{z} + b_{30}z^3 + b_{03}\bar{z}^3 + b_{21}z^2\bar{z} + b_{12}z\bar{z}^2),$$

where $b_{jk} \in \mathbb{R}$, or equivalently

$$\begin{aligned}
 \dot{x} &= -(b_{10} - b_{01})y + (b_{21} + b_{03} - b_{30} + b_{12})y^3 \\
 &\quad + (3(b_{30} - b_{03}) + b_{21} - b_{12})yx^2 \\
 \dot{y} &= (b_{10} + b_{01})x + (b_{30} + b_{03} + b_{12} + b_{21})x^3 \\
 &\quad + (b_{21} + b_{12} - 3(b_{30} + b_{03}) + b_{21} - b_{12})y^2x
 \end{aligned}
 \tag{75}$$

By introducing the following notations

$$\begin{aligned}
 b_{01} &= \frac{1}{2}(\alpha + a), \quad b_{10} = \frac{1}{2}(\alpha - a), \\
 b_{03} &= \frac{1}{8}(\beta + b - \gamma - c), \quad b_{30} = \frac{1}{8}(\beta + c - \gamma - b), \\
 b_{12} &= \frac{1}{8}(\gamma + b + 3(\beta + c)), \quad b_{21} = \frac{1}{8}(\gamma - b + 3(\beta - c)),
 \end{aligned}$$

we obtain from (75) the cubic planar vector field

$$\dot{x} = y(a + bx^2 + cy^2), \quad \dot{y} = x(\alpha + \beta x^2 + \gamma y^2).
 \tag{76}$$

We assume that

$$c(b\gamma - c\beta)((b - \gamma)^2 + 4c\beta) \neq 0.
 \tag{77}$$

It is easy to prove that system (76) is invariant under the change $(x, -y, -t) \rightarrow (x, y, t)$ and under the change $(-x, y, -t) \rightarrow (x, y, t)$.

The following corollary was proved in [30, 17].

Corollary 57. *System (76) under the condition (77) has the following properties:*

- (i) *admits two invariant conics (eventually imaginary)*

$$g_j = \nu_j(x^2 - \lambda_1) - (y^2 - \lambda_2) = 0, \quad \text{for } j = 1, 2,$$

where ν_1 and ν_2 are the roots of the polynomial $P(\nu) = c\nu^2 + (b - \gamma)\nu - \beta$, and

$$\lambda_1 = \frac{\gamma a - \alpha c}{b\gamma - c\beta}, \quad \lambda_2 = \frac{\alpha b - \beta a}{b\gamma - c\beta}.$$

In view of (77) we obtain that $\nu_1 - \nu_2 \neq 0$,

- (ii) *admits the first integral*

$$F(x, y) = \frac{(\nu_1(x^2 - \lambda_1) - y^2 + \lambda_2)^{\gamma - \nu_2 c}}{(\nu_2(x^2 - \lambda_1) - y^2 + \lambda_2)^{\gamma - \nu_1 c}};$$

(iii) the solutions can be represented as follows

$$\begin{aligned} x^2 &= \lambda_1 + X(\tau, x_0, y_0) \\ y^2 &= \lambda_2 + Y(\tau, x_0, y_0) \\ t &= t_0 + \int_0^\tau \frac{d\tau}{\sqrt{(\lambda_1 + X(\tau, x_0, y_0))(\lambda_2 + Y(\tau, x_0, y_0))}} \end{aligned}$$

where X, Y are solutions of the linear differential equation of the second order with constants coefficients

$$T'' - (\gamma + b)T' + (b\gamma - c\beta)T = 0, \quad \text{where } ' \equiv \frac{d}{d\tau}$$

Proposition 58. *Differential system (76) is Poincaré–Liapunov integrable if and only $a\alpha < 0$.*

Proof. We shall study the following three cases.

First we assume that (77) holds and

$$\Lambda = 2(\nu_1 - \nu_2) (\lambda_2 - \nu_1 \lambda_1)^{\gamma - \nu_2 c - 1} (\lambda_2 - \nu_2 \lambda_1)^{\gamma - \nu_2 c - 1} \neq 0.$$

By developing the function

$$G(x, y) = \frac{(\nu_1(x^2 - \lambda_1) - y^2 + \lambda_2)^{\gamma - \nu_2 c}}{(\nu_2(x^2 - \lambda_1) - y^2 + \lambda_2)^{\gamma - \nu_1 c}}$$

in Taylor series we obtain

$$\frac{G}{\Lambda} = (\alpha x^2 - ay^2 + \kappa_1 x^4 + \kappa_2 y^4 + \dots)$$

and κ_1, κ_2 are convenient constants. Hence, by supposing that $a > 0$ we get

$$\alpha x^2 - ay^2 + \kappa_1 x^4 + \kappa_2 y^4 + \dots = a\Lambda (-qx^2 + y^2 + \kappa_1 x^4 + \kappa_2 y^4 + \dots) > 0,$$

in the neighborhood of the origin where $q = \frac{\alpha}{a} < 0$.

Second We assume that $c = 0$ and $\gamma(\gamma - b) \neq 0$. For simplicity we shall assume that $a = 1$, $\alpha = -1$ and $b > 0$. After some computations we can prove that the function

$$F = \frac{y^2 + \frac{-(\gamma - b) + \beta(a + \gamma x^2)}{\gamma^2 - \gamma b}}{(1 + bx^2)^{\gamma/b}},$$

is a first integral of the system

$$\dot{x} = y(1 + bx^2), \quad \dot{y} = x(-1 + \beta x^2 + \gamma y^2).$$

By developing the function F in Taylor series in the neighborhood of the origin we obtain that

$$F = 2(x^2 + y^2 + \kappa_1 x^4 + \kappa_2 y^4 + \dots),$$

where κ_1 and κ_2 are conveniens constants.

Now we study the case when $c = 0$ and $b = \gamma \neq 0$. The function

$$F = \frac{y^2 - \frac{1}{b} + \frac{\beta}{b^2}}{1 + bx^2} - \frac{\beta}{b^2} \log(1 + bx^2),$$

is a first integral of the system

$$(78) \quad \dot{x} = y(1 + bx^2), \quad \dot{y} = x(-1 + \beta x^2 + by^2).$$

By developing the function F in Taylor series in the neighborhood of the origin we obtain that

$$F = \frac{2}{1}(x^2 + y^2 + \kappa_1 x^4 + \kappa_2 y^4 + \dots),$$

where κ_1 and κ_2 are conveniens constants.

Finally, For the case when $c = 0$ and $\gamma = 0$ we have that the function

$$F = y^2 - \frac{\beta}{b}x^2 + \frac{(\beta + b)}{b^2} \log(1 + bx^2),$$

is a first integral of the system

$$\dot{x} = y(1 + bx^2), \quad \dot{y} = x(-1 + \beta x^2).$$

The Taylor expansion of F in the neighborhood of the origin is

$$F = \frac{2}{a}(x^2 + y^2 + \kappa_1 x^4 + \kappa_2 y^4 + \dots),$$

where κ_1 and κ_2 are conveniens constants. Thus the proposition is proved. \square

Now we shall study the particular case when (76) is such that

$$(79) \quad \begin{aligned} \dot{x} &= y(\lambda b + p + x^2(\lambda + b - 2a) + y^2(\lambda - b)), \\ \dot{y} &= x(-\lambda a - p - x^2(\lambda - a) - y^2(\lambda + a - 2b)), \end{aligned}$$

where λ, b, a, p are real parameters and $b - a \neq 0$. Then it is easy to obtain that

$$\nu_1 = -\frac{\lambda - a}{\lambda - b}, \quad \nu_2 = -1.$$

Thus the invariant curves of the differential system are

$$g_1 = -(y^2 + x^2 + \lambda) = 0, \quad g_2 = -\left((\lambda - a)x^2 + (\lambda - b)y^2 + \frac{1}{2}(\lambda^2 + p)\right) = 0.$$

The first integral F takes the form

$$F(x, y) = \frac{(y^2 + x^2 + \lambda)^2}{(\lambda - a)x^2 + (\lambda - b)y^2 + \frac{1}{2}(\lambda^2 + p)}.$$

Consequently all trajectories of (79) are algebraic curves

$$(80) \quad (x^2 + y^2)^2 + A(K)x^2 + B(K)y^2 + P(K) = 0,$$

where $F(x, y) = K$ are the level curves, and

$$A(K) = 2(\lambda - \frac{K}{2}(\lambda - a)), \quad B(K) = 2(\lambda - \frac{K}{2}(\lambda - b)), \quad P(K) = \lambda^2 - \frac{1}{2}K(\lambda^2 + p).$$

It is interesting to observe that a particular case of the family of planar curves, which is the locus of point (x, y) the product of whose distance from the fixed points $(0, -c)$ and $(0, c)$ has the constant value $\kappa^2 - c^2$ (for more details see [4]). The quartic equation of this curve is

$$(81) \quad (x^2 + y^2)^2 + 2c^2(x^2 - y^2) = \kappa^2(\kappa^2 - 2c^2),$$

which is equivalent to

$$(82) \quad (z^2 + c^2)\overline{(z^2 + c^2)} = (\kappa^2 - c^2)^2.$$

Thus first, if $A(K) = -B(K) = 2c^2$, and $P(K) = -\kappa^2(\kappa^2 - 2c^2)$, then if $K \neq 2$, we obtain that

$$K = \frac{4c^2}{a - b}, \quad p = \frac{(a - b)(\kappa^2 - c^2)^2}{2c^2} + \frac{(c^2 - 2ab)c^2}{a - b - 2c^2}, \quad \lambda = \frac{(a + b)c^2}{2c^2 - a - b},$$

and second, if $K = 2$ then

$$A(2) = 2a, \quad B(2) = 2b, \quad P(2) = -p, \quad a = -b = c^2, \quad p = \kappa^2(\kappa^2 - 2c^2),$$

for arbitrary λ .

For the first case system (79) takes the form

$$\begin{aligned} x' &= -y((a - b)((2a - b - 3c^2)x^2 + (c^2 + b)y^2) \\ &\quad + p(2c^2 + b - a) + c^2b(a + b)), \\ y' &= x((a - b)((a - c^2)x^2 + (3c^2 + 2b - a)y^2) \\ &\quad + p(2c^2 + b - a) + c^2a(a + b)), \end{aligned}$$

where $' = \frac{d}{d\tau}$, with $t = (a - b - 2c^2)\tau$. This differential system admits as trajectories the family of lemniscate (81).

For the second case we obtain that the differential system (79) in complex coordinates, takes the form

$$\dot{z} = i \left(\kappa^2 (2c^2 - \kappa^2) z + c^2 z^3 - \lambda \bar{z} (c^2 + z^2) \right).$$

and admits the family of lemniscate (82). In particular if $c = 1$ then we obtain the system

$$\dot{z} = i \left((1 - C^2) z + z^3 - \lambda \bar{z} (1 + z^2) \right).$$

The bifurcation diagrams of this differential system in the plane ($C = |\kappa^2 - 1|, \lambda$) are given in [30]. Now we study the case when $\lambda = 0$, i.e. we shall study the differential equation for which the origin is holomorphic center

$$\dot{z} = i \left((1 - C^2) z + z^3 \right).$$

The trajectories of this equation are given by the formula (80) and are the lemniscates given by

$$(z^2 + 1)(\bar{z}^2 + 1) = (\kappa^2 - 1)^2 = C^2.$$

Now we study the following particular case of system (76)

$$\dot{x} = y \left(a + (r - q)x^2 + y^2 \right), \quad \dot{y} = x \left(\alpha - (p^2 + q^2)x^2 + (r + q)y^2 \right).$$

It is easy to show that the roots ν_1 and ν_2 are $\nu_1 = q + ip$ and $\nu_2 = q - ip$. Thus the invariant curves are complex

$$g_1 = (q + ip)(x^2 - \lambda_1) - (y^2 - \lambda_2) = 0, \quad g_2 = (q - ip)(x^2 - \lambda_1) - (y^2 - \lambda_2) = 0.$$

Hence the first integral F is

$$F(x, y) = ((q(x^2 - \lambda_1) - (y^2 - \lambda_2))^2 + p^2(x^2 - \lambda_1)^2) \cdot \exp 2 \arctan \left(\frac{p(x^2 - \lambda_1)}{q(x^2 - \lambda_1) - (y^2 - \lambda_2)} \right).$$

Particular case of (74) is

$$\dot{z} = i \left(z + b_{20}z^2 + b_{30}z^3 \right),$$

or equivalently

$$\begin{aligned} \dot{x} &= -y - 2b_{20}xy + 3b_{30}yx^2 - 3b_{30}y^3, \\ \dot{y} &= x + b_{20}x^2 - b_{20}y^2 + b_{30}x^3 - 3b_{30}y^2x, \end{aligned}$$

clearly the origin is holomorphic center for this cubic system,

It is well known the following results (see for [18]).

Theorem 59. *The quasi-homogenous cubic differential system (70) has an isochronous center at the origin if and only if a linear change of coordinates x, y and a scaling of time t transform cubic differential system to one of the systems*

$$\begin{aligned}
 & i) \quad \dot{x} = y(1 + x^2), \quad \dot{y} = -x(1 - y^2), \\
 & ii) \quad \dot{x} = y(1 - 3x^2 + y^2), \quad \dot{y} = -x(1 + 3y^2 - x^2) \\
 (83) \quad & \iff \dot{z} = i(z + z^3), \\
 & iii) \quad \dot{x} = y(1 + 9x^2 - 2y^2), \quad \dot{y} = -x(1 - 3y^2), \\
 & iv) \quad \dot{x} = y(1 - 9x^2 + 2y^2), \quad \dot{y} = -x(1 + 3y^2).
 \end{aligned}$$

The following result was proved in [20]

Theorem 60. *The origin is an isochronous center of (70) if and only if one of the following sets of conditions is satisfied: (i)*

$$A + C = 0, \quad A - L = 0, \quad A + N = 0, \quad B - M = 0, \quad D = K = 0,$$

(ii)

$$\begin{aligned}
 3A + C &= 0, & 3A - L &= 0, & A + N &= 0, & B + 3D &= 0, \\
 B + 3K &= 0, & B - M &= 0,
 \end{aligned}$$

(iii)

$$\begin{aligned}
 3A + L + C + 3N &= 0, & 9A - 5L + 2C - 9N &= 0, \\
 B + 3D - 3K - M &= 0, & B + 6D + 6K + M &= 0, \\
 (3A + 7N)(7A + 3N) - 100DK &= 0, \\
 (A + N)((3A + L)^2 - (B + M)^2) - 2(3A + L)(B + M)(D - K) &= 0
 \end{aligned}$$

Remark 61. *From the previous results we have the following remarks.*

- (i) *From Proposition 59 follows that any cubic quasi-homogenous differential system with isochronous center at the origin under linear change of coordinates x, y and a scaling of time t can be transformed any reversible cubic system of the type (83) which is a particular case of the system (74).*
- (ii) *The following question arise, It is possible to show that under linear change of coordinates x, y and a scaling of time t to transform cubic non-reversible differential system (72) with two isochronous center and with $N \neq 0$ in to the one of the reversible cubic quasi-homogenous differential system (83) ? If the answer is negative then Theorem 59 and 60 are not equivalent.*

- (iii) *System (70) under the conditions (ii) of Theorem 60 in complex coordinates becomes*

$$\dot{z} = iz + \left(A - i\frac{B}{3}\right)z^3,$$

which coincide with the system (72) if we take $N = -A$ and $r = \frac{B}{6}$.

- (iv) *In [13] the authors proposed the following results.*

Theorem 62. *The origin is an isochronous center of the cubic system*

$$\dot{z} = iz + Dz^3 + Ez^2\bar{z} + Fz\bar{z}^2 + G\bar{z}^3,$$

if and only if one of the following three relations is satisfied

$$b1 : \quad E = F = G = 0,$$

$$b2 : \quad D = \bar{F}, \quad E = G = 0,$$

$$b3 : \quad D = \frac{7}{3}\bar{F}, \quad E = 0, \quad |G|^2 = \frac{16}{9}|F|^2, \quad \Re(\bar{F}^2G) = 0$$

The solutions of conditions b3 produces the following cubic systems

- (1) *The cubic system which satisfy condition b3) are the following.*

$$\dot{x} = -y\left(1 + \frac{3K}{2}x^2\right), \quad \dot{y} = x\left(1 + K\left(x^3 - \frac{9}{2}y^2\right)\right),$$

which is a particular case of (78).

- (2) *and the cubic system*

$$\begin{aligned} \dot{x} = & -y + \mu(10Ax^3 - 6(N + 4A)xy^2) \\ & + (N + 9A)(7N + 3A)y^3 + (9N^2 - 198NA - 111A^2)xy^2, \\ \dot{y} = & x + \mu\left(6(4N + A)x^2y - \frac{90NA}{N + 9A}y^3\right) \\ & + (3N + 7A)(A + 9N)x^3 - (111N^2 + 9A^2 - 198AN)xy^2 \end{aligned}$$

$$\text{where } \mu = \sqrt{-\frac{A + 9N}{N + 9A}} \in \mathbb{R}, \text{ satisfy the conditions b3).}$$

10. QUARTIC DIFFERENTIAL SYSTEM WITH NON-DEGENERATE CENTER

In this section we solve the inverse problem of the center for quartic planar differential equations.

Proposition 63. *Quartic differential system*

$$\begin{aligned}
 \dot{x} &= -y + ax^2 + by^2 + cxy + Ax^3 + Bx^2y + Cxy^2 + Dy^3 \\
 &\quad + L_1x^4 + L_2x^3y + L_3x^2y^2 + L_4xy^3 + L_5y^4 \\
 &:= -y + X, \\
 \dot{y} &= x + \alpha x^2 + \beta y^2 + \gamma xy + Kx^3 + Lx^2y + Mxy^2 + Ny^3 \\
 &\quad + K_1x^4 + K_2yx^3 + K_3x^2y^2 + K_4x^3y + K_5y^4 \\
 &:= x + Y,
 \end{aligned} \tag{84}$$

can be rewritten as follows

$$\begin{aligned}
 \dot{x} &= \{H_2 + H_3 + H_4 + H_5, x\} + g\{H_2, x\}, \\
 \dot{y} &= \{H_2 + H_3 + H_4 + H_5, y\} + g\{H_2, y\},
 \end{aligned} \tag{85}$$

if and only if

$$\int_0^{2\pi} \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) \big|_{x=\cos t, y=\sin t} dt = 3(N + A) + L + C = 0$$

where $g = g(x, y)$ is a polynomial of degree 3 and $H_j = H_j(x, y)$ is a homogenous polynomial of degree j for $j = 2, 3, 4, 5$.

Proof. Indeed, under the given conditions on the parameters, the functions H_3, H_4, H_5 and g are determined as follows

$$\begin{aligned}
 H_3 &= \frac{c + \alpha + 2k_{21}}{3} x^3 - \frac{\gamma + b + 2a}{3} y^3 - ax^3y + k_{21}xy^2, \\
 H_4 &= \frac{M + B + K + D}{4} y^4 - \frac{B + K}{2} x^2y^2 - Ax^3y \\
 &\quad - \frac{L + 3A + C}{3} xy^3 - \Lambda(x^2 + y^2)^2, \\
 H_5 &= \frac{2K_3 + 3(K_1 + L_2) + 2L_4 + 8L_1 + 2K_2}{15} x^5 - \frac{3(L_5 + K_4) + 8L_1 + 2(K_2 + L_3)}{15} y^5 \\
 &\quad + K_5xy^4 - L_1x^4y - \frac{4L_1 + K_2 + L_3}{3} x^2y^3 + \frac{4K_5 + K_3 + L_4}{3} x^3y^2, \\
 g &= -(2k_{21} + c)x + (\gamma + 2a)y + By^2 + (L + 3A)xy + (K - 4\Lambda)(x^2 + y^2) \\
 &\quad (4L_1 + K_2)yx^2 - (4K_5 + L_4)xy^2 - \frac{(2K_3 + 2L_4 + 3L_2 + 8K_5)}{3} x^3 \\
 &\quad + \frac{3K_4 + 2L_3 + 8L_1 + 2K_2}{3} y^3,
 \end{aligned}$$

where Λ is an arbitrary constant. □

Corollary 64. *Differential system (84) is Hamiltonian if and only if $g = (K - 4\Lambda_2)(x^2 + y^2)$ (see Corollary 15) i.e.*

$$\begin{aligned}
 (86) \quad & 2k_{21} + c = 0, \quad \gamma + 2a = 0, \quad 3K_4 + 2L_3 + 8L_1 + 2K_2 = 0, \\
 & 4L_1 + K_2 = 0, \quad 3(N + A) + L + C = 0, \quad 4K_5 + L_4 = 0, \\
 & L + 3A = 0, \quad 2K_3 + 2L_4 + 3L_2 + 8K_5 = 0,
 \end{aligned}$$

Proof. From the divergence condition we get that $\{H_2, g\} = 0$ follows $g = g(H_2)$. In view of (86) we deduce that $g = (K - \Lambda_2)(x^2 + y^2)$. Thus differential system (85) becomes

$$\begin{aligned}
 \dot{x} &= \{H_2 + \tilde{H}_3 + \tilde{H}_4 + \tilde{H}_5, x\} + 2(K - \Lambda_2)H_2\{H_2, x\} \\
 &= \{H_2 + \tilde{H}_3 + \tilde{H}_4 + \tilde{H}_5 + (K - \Lambda_2)H_2^2, x\} := \{H, x\} \\
 \dot{y} &= \{H_2 + \tilde{H}_3 + \tilde{H}_4 + \tilde{H}_5, y\} + 2(K - \Lambda_2)H_2\{H_2, y\} \\
 &= \{H_2 + \tilde{H}_3 + \tilde{H}_4 + \tilde{H}_5 + (K - \Lambda_2)H_2^2, y\} := \{H, y\},
 \end{aligned}$$

where \tilde{H}_j correspond to the value of H_j under the conditions (86), for $j = 3, 4, 5$. Hence the system is Hamiltonian with Hamiltonian $H = H_2 + \tilde{H}_3 + \tilde{H}_4 + \tilde{H}_5 + (K - \Lambda_2)H_2^2$.

□

Proposition 65. *The quartic differential system*

$$\begin{aligned}
\dot{x} = & -y + ax^2 + \frac{a(3\lambda - 2)}{\lambda}y^2 - \frac{2\kappa(\lambda - 1)}{\lambda}xy \\
& Ax^3 + Bx^2y + \frac{A(3\lambda - 2)}{\lambda}xy^2 + L_1x^4 + L_3x^2y^2 \\
& + \frac{((4\lambda^2 - 6\lambda + 2)\Lambda + \lambda(2\lambda - 1)B)}{\lambda^2}y^3 - \frac{2K_5(2\lambda - 1)}{\lambda}xy^3 \\
& - \frac{(15\lambda^2 - 16\lambda + 4)L_1 - \lambda(5\lambda - 2)L_3}{3\lambda^2}y^4 \\
& - \frac{2((5\lambda - 2)K_5 + \lambda(\lambda - 1)K_3)}{3\lambda^2}x^3y, \\
\dot{y} = & x + \frac{\kappa(3\lambda - 2)}{\lambda}x^2 + \kappa y^2 - \frac{2a(\lambda - 1)}{\lambda}xy \\
& + \frac{2\Lambda(2\lambda - 1)}{\lambda}x^3 - \frac{A(3\lambda - 2)}{\lambda}x^2y \\
& - \frac{(\lambda(\lambda - 1)B - 2(2\lambda - 1)\Lambda)}{\lambda^2}xy^2 - Ay^3 + K_5y^4 \\
& + K_3x^2y^2 - \frac{\lambda(\lambda - 1)L_3(5\lambda - 2)L_1}{3\lambda^2}xy^3 - \frac{2(2\lambda - 1)L_1}{\lambda}x^3y \\
& - \frac{(16\lambda - 4 - 15\lambda^2)K_5 + \lambda(5\lambda - 2)K_3}{3\lambda^2}x^4
\end{aligned}$$

where κ and Λ are arbitrary constants, is Poincaré–Liapunov integrable if $\lambda \in \mathbb{R} \setminus \{0\}$.

Proof. Indeed, after some computations we can prove that the function $F = (1 + (1 - \lambda)g)H_2^{(\lambda-1)/\lambda}$, is a first integral if $\lambda \in \mathbb{R} \setminus \{0, 1\}$, where

$$\begin{aligned}
g = & -\frac{2((3\lambda - 2)L_1 - \lambda L_3)}{3\lambda^2}y^3 - \frac{2((2 - 3\lambda)K_5 + \lambda K_3)}{3\lambda^2}x^3 \\
& \frac{2L_1}{\lambda}x^2y - \frac{2K_5}{\lambda}xy^2 - \frac{2\Lambda}{\lambda}x^2 + \frac{(2\lambda - 2)\Lambda + \lambda B}{\lambda^2}y^2 \\
& \frac{2A}{\lambda}xy - \frac{2\kappa}{\lambda}x + \frac{2a}{\lambda}y,
\end{aligned}$$

and $F = H_2e^{-\tilde{g}}$ if $\lambda = 1$, where $\tilde{g} = g|_{\lambda=1}$. □

Remark 66. *After tedious computations it is possible to show that any differential system (84) can be rewritten as follows*

$$\begin{aligned}\dot{x} &= \{H_2 + H_3 + H_4 + H_5, x\} + g_1\{H_2 + H_3 + H_4, x\} \\ &\quad + g_2\{H_2 + H_3, x\} + g_3\{H_2, x\}, \\ \dot{y} &= \{H_2 + H_3 + H_4 + H_5, y\} + g_1\{H_2 + H_3 + H_4, y\} \\ &\quad + g_2\{H_2 + H_3, y\} + g_3\{H_2, y\},\end{aligned}$$

11. QUARTIC QUASI-HOMOGENOUS VECTOR FIELD WITH NON-DEGENERATE CENTER

We shall study the polynomial planar differential system of degree four of the type

$$\begin{aligned}\dot{x} &= -y + L_{40}x^4 + L_{04}y^4 + L_{22}x^2y^2 + L_{13}xy^3 + L_{31}x^3y \\ &:= -y + X, \\ \dot{y} &= x + K_{40}x^4 + K_{04}y^4 + K_{22}x^2y^2 + K_{13}xy^3 + K_{31}x^3y \\ &:= x + Y,\end{aligned}\tag{87}$$

Proposition 67. *System (87) can be rewritten as*

$$\begin{aligned}\dot{x} &= -y - \frac{\partial H}{\partial y} - yg, \\ \dot{y} &= x + \frac{\partial H}{\partial x} + xg,\end{aligned}\tag{88}$$

where $H = H_3 + H_4 + H_5$, and $g = g(x, y)$ is a convenient polynomial of degree three.

Proof. Indeed, in this case

$$\int_0^{2\pi} \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) \Big|_{x=\cos t, y=\sin t} dt \equiv 0,$$

where X and Y are polynomials given by the formula (87). In this case the function H and g for which (88) holds are

$$\begin{aligned}
H = & \Lambda(x^2 + y^2)^2 + \frac{1}{15} (8K_{04} + 2K_{22} + 2L_{13} + 3K_{40} + 3L_{31}) x^5 \\
& + K_{04}xy^4 - L_{40}yx^4 - \frac{1}{15} (8L_{40} + 2L_{22} + 2K_{31} + 3L_{04} + 3K_{13}) y^5 \\
& - \frac{1}{3} (4L_{40} + L_{22} + K_{31}) x^2y^3 + \frac{1}{3} (4K_{04} + K_{22} + L_{13}) x^2y^3, \\
g = & -4\Lambda(x^2 + y^2) - \frac{1}{3} (8K_{04} + 2K_{22} + 2L_{13} + 3L_{31}) x^3 + (4L_{40} + K_{31})x^2y \\
& + \frac{1}{3} (8L_{40} + 2L_{22} + 2K_{31} + 3K_{13}) y^3 - (4K_{04} + L_{13})xy^2
\end{aligned}$$

□

Example 68. *Quartic differential system*

$$(89) \quad \dot{x} = -y + y^4, \quad \dot{y} = x + x^4 - x^2y^2,$$

admits a center at the origin and foci at the point $(\approx -1, 3247, 1)$.

Proposition 69. *Quartic quasi homogenous differential system*

$$\begin{aligned}
\dot{x} = & -y + L_{40}x^4 + L_{22}x^2y^2 - \frac{2K_{04}}{\lambda}(2\lambda - 1)xy^3 \\
& - \frac{1}{3\lambda^2} ((10\lambda - 4)K_{04} + 2\lambda(\lambda - 1)K_{22}) x^3y \\
& - \frac{1}{3\lambda^2} (\lambda(2 - 5\lambda)L_{22} + ((15\lambda^2 - 16\lambda + 4)L_{40}) y^4, \\
(90) \quad \dot{y} = & x + K_{04}y^4 + K_{22}x^2y^2 - \frac{2(2\lambda - 1)L_{40}}{\lambda}x^3y \\
& - \frac{2}{3\lambda^2} (5\lambda - 2)L_{40} + \lambda(\lambda - 1)L_{22}) xy^3 \\
& - \frac{2}{3\lambda^2} ((15\lambda^2 - 16\lambda + 4)K_{04} + (2\lambda - 5\lambda^2)K_{22}) x^4
\end{aligned}$$

is Poincaré–Liapunov integrable.

Proof. It is possible to show that the function $F = (1 + (1 - \lambda))H^{(\lambda-1)/\lambda}$ for $\lambda \in \mathbb{R} \setminus \{0\}$ with

$$\begin{aligned}
g = & \frac{2}{3\lambda^2} ((3\lambda - 2)K_{04} - \lambda K_{22}) x^3 + \frac{2}{3\lambda^2} (2(L_{40}(2 - 3\lambda) + \lambda L_{22}) y^3 \\
& + \frac{2L_{40}}{\lambda}x^2y + -\frac{2K_{04}}{\lambda}xy^2,
\end{aligned}$$

is a first integral of (90) and if $\lambda = 1$ then $F = H_2 e^{-\tilde{g}}$ with $\tilde{g} = g|_{\lambda=1}$ is a first integral of (90) with $\lambda = 1$, i.e. is a first integral of quartic system

$$\begin{aligned}\dot{x} &= -y + (x^2 + y^2)(-L_{40}x^2 - (L_{22} - L_{40})y^2 + 2K_{04}xy) \\ \dot{y} &= x - (x^2 + y^2)((K_{04} - K_{22})x^2 + 2L_{40}xy - K_{04}y^2).\end{aligned}$$

Thus the proof follows. \square

Corollary 70. *The quartic quasi-homogenous differential system*

$$\begin{aligned}\dot{x} &= -y + x(L_{40}x^3 + L_{22}xy^2 + K_{04}y^3 + K_{22}x^2y), \\ \dot{y} &= x + y(L_{40}x^3 + L_{22}xy^2 + K_{04}y^3 + K_{22}x^2y).\end{aligned}$$

has an uniformly isochronous center at the origin and admits the first integral

$$F = \frac{((K_{22} + 2K_{04})x^3 - 3L_{40}x^2y + 3K_{04}xy^2 - (L_{22} + 2L_{40})y^3 - 1))}{(x^2 + y^2)^3}.$$

Proof. Follows from Corollary 18 with $m = 4$. \square

Corollary 71. *Differential system (87) is Hamiltonian if and only if*

$$(91) \quad \begin{aligned}L_{22} + 3/2K_{13} &= 0, & 4L_{40} + K_{31} &= 0, \\ 4K_{40} + K_{22} + L_{13} + 3/2L_{31} &= 0.\end{aligned}$$

Proof. From the previous result follows that differential system (87) can be rewritten as follows

$$\dot{x} = -y - \frac{\partial H_5}{\partial y}, \quad \dot{y} = x + \frac{\partial H_5}{\partial x},$$

if and only if $g_3 \equiv 0$, which is equivalent to (91). \square

Proposition 72. *The quartic differential system*

$$\begin{aligned}\dot{x} &= -y - \frac{(5\lambda - 2)((3\lambda - 2)L_{40} - \lambda L_{22})}{3\lambda^2}y^4 + L_{40}x^4 + L_{22}x^2y^2 \\ &\quad - \frac{2(\lambda(\lambda - 1)K_{22} + (2 - 5\lambda)L_{40})}{3\lambda^2}yx^3, \\ \dot{y} &= x + \frac{(5\lambda - 2)((3\lambda - 2)L_{40} + \lambda K_{22})}{3\lambda^2}x^4 - L_{40}y^4 + K_{22}x^2y^2 \\ &\quad - \frac{2(2\lambda - 1)L_{40}}{\lambda}yx^3 - \frac{2(\lambda(\lambda - 1)L_{22} + (5\lambda - 2)L_{40})}{3\lambda^2}xy^3\end{aligned}$$

with $\lambda \in \mathbb{R} \setminus \{0, 1\}$ admits the Darboux first integral

$$F = \left(1 + (1 - \lambda)g_3\right)H_2^{(\lambda-1)/\lambda}$$

where

$$g_3 = \frac{(\lambda K_{22} + (3\lambda - 2)L_{40})}{3\lambda^2} x^3 + \frac{2L_{40}}{\lambda} xy(x+y) - \frac{((3\lambda - 2)L_{40} - \lambda L_{22})}{3\lambda^2} y^3.$$

and if $\lambda = 1$ the quartic quasi homogenous differential system

$$\begin{aligned}\dot{x} &= -y + (x^2 + y^2)((L_{40}(x^2 + 2xy - y^2) + L_{22}y^2), \\ \dot{y} &= x + (x^2 + y^2)((L_{40}(x^2 - 2xy - y^2) + K_{22}x^2),\end{aligned}$$

admits the first integral $H_2 e^{-\tilde{g}_3} = \text{Const.}$ where $\tilde{g}_3 = g_3|_{\lambda=1}$.

Proof. From the equation

$$H_5 + \lambda g_3 H_2 = \sum_{j+k=5} a_{jk} x^j y^k = 0$$

and by solving $a_{jk} = 0$ for $j + k = 5$ we deduce the proof of the proposition. \square

In [33] the following theorem was proved (see Theorem 2)

Theorem 73. *Differential system*

$$(92) \quad \begin{aligned}\dot{x} &= -y, \\ \dot{y} &= x + K_{40}x^4 + K_{04}y^4 + K_{22}x^2y^2 + K_{13}xy^3 + K_{31}x^3y,\end{aligned}$$

or equivalently

$$\dot{z} = iz + A_{40}z^4 + A_{04}\bar{z}^4 + A_{22}z^2\bar{z}^2 + A_{13}z\bar{z}^3 + A_{31}z^3\bar{z},$$

is called the quartic lopsided system where A_{jk} are the following parameters

$$\begin{aligned}A_{40} &= \frac{1}{16} (K_{40} + K_{04} - K_{22} + i(K_{31} - K_{13})), \\ A_{04} &= \frac{1}{16} (K_{40} + K_{04} - K_{22} - i(K_{31} - K_{13})), \\ A_{31} &= \frac{1}{8} (2K_{40} - 2K_{04} + i(K_{31} + K_{13})), \\ A_{13} &= \frac{1}{8} (2K_{40} - 2K_{04} - i(K_{31} + K_{13})), \\ A_{22} &= \frac{1}{8} (3K_{40} + 3K_{04} + K_{22}),\end{aligned}$$

The origin is a center for the the lopsided differential system if and only if one of the following conditions hold

- (1) All the coefficients A_{jk} with $j + k = 4$ are reals.
- (2) $A_{13} = A_{22} = 0$ and $\Re(A_{04}) = 0$.
- (3) $A_{04} = A_{22} = 0$ and $\Re(A_{13}) = 0$.

- (3) $A_{22} = \Re(A_{13}) = \Re(A_{04}) = 0$ and $\Im(A_{04}) + \beta\Im(A_{13}) = 0$, where $\beta \in \mathbb{R}$.

After some computations from the given conditions we get

- (1) $K_{13} = K_{31} = 0$, thus differential system (92) becomes

$$\dot{x} = -y, \quad \dot{y} = x + K_{04}y^4 + K_{40}x^4 + K_{22}x^2y^2.$$

- (2)

$$K_{04} = K_{40} = K_{22} = 0, \quad K_{13} = -K_{31},$$

thus differential system (92) becomes

$$(93) \quad \dot{x} = -y, \quad \dot{y} = x + K_{13}xy(x^2 - y^2).$$

- (3)

$$K_{04} = K_{40} = K_{22} = 0, \quad K_{13} = K_{31},$$

thus differential system (92) becomes

$$\dot{x} = -y, \quad \dot{y} = x + K_{31}xy(x^2 + y^2).$$

- (4) a)

$$K_{04} = K_{40} = 0, \quad K_{13} = \frac{2b-1}{1+2b}K_{31}, \quad K_{22} = 0, \quad K_{40} = 0,$$

with $\beta \neq -1/2$ thus differential system (92) becomes

$$\dot{x} = -y, \quad \dot{y} = x + K_{31} \left(x^3y - \frac{2b-1}{1+2b}xy^3 \right).$$

- b) If $\beta = -1/2$ then

$$K_{04} = K_{22} = K_{31} = K_{40} = 0,$$

thus differential system (92) becomes

$$\dot{x} = -y, \quad \dot{y} = x + K_{13}xy^3.$$

We observe that the differential systems (93) are particular case of the system

$$\dot{x} = -y, \quad \dot{y} = x + ax^4 + by^4 + cx^2y^2,$$

where a, b and c are arbitrary constants, which is invariant under the change $(x, -y, -t) \longrightarrow (x, y, t)$ and the differential systems of the other cases are particular cases of the differential system

$$\dot{x} = -y, \quad \dot{y} = x + xy(\alpha x^2 + \beta y^2),$$

where α, β and γ are arbitrary constants, which is invariant under the change $(-x, y, -t) \longrightarrow (x, y, t)$ (see Proposition 9).

Corollary 74. *Theorem 73 give only sufficient conditions.*

Proof. Indeed the following non-reversible quartic differential system

$$(94) \quad \dot{x} = -y, \quad \dot{y} = x + ay(y^3 - 4x^3),$$

has a center at the origin .

□

Corollary 75. *Quartic quasihomogenous differential system with holomorphic center is*

$$(95) \quad \begin{aligned} \dot{x} &= -y + L_{40}x^4 + L_{22}x^2y^2 - 4L_{40}xy^3 - (5L_{40} + L_{22})y^4 \\ &\quad + (2K_{22} - 8L_{40})yx^3, \\ \dot{y} &= x + (5L_{40} - K_{22})x^4 + 4L_{40}yx^4 + K_{22}x^2y^2 \\ &\quad - L_{40}y^4 + (2L_{22} + 8L_{40})xy^3. \end{aligned}$$

is Poincaré–Liapunov integrable with the first integral

$$\frac{1 + 2(5L_{40} - K_{22})x^3 + 2(5L_{40} + L_{22})y^3 + 6L_{40}xy(x + y)}{H_2^3} = \text{Const..}$$

Proof. Follows from Proposition 72 with $\lambda = 1/4$. We observe that (95) by introducing the respectively notations, coincide with the quartic planar differential system deduced in [10].

□

Corollary 76. *The quasi-homogenous quartic planar differential system with holomorphic center is*

$$\begin{aligned} \dot{x} &= -y + L_{40}(x^4 + y^4 - 6x^2y^2 - 4xy^3 + 4yx^3), \\ \dot{y} &= x + L_{40}(-y^4 - x^4 + 6x^2y^2 - 4xy^3 + 4yx^3), \end{aligned}$$

or equivalently

$$\dot{z} = iz + L_{40}(1 - i)z^4.$$

Proof. By solving the equations

$$\Delta H_5 + 2g_3 = 0, \quad H_2g + 4H_5 = 0,$$

we obtain the proof.

We observe that the first integral in this case is

$$F = \frac{1 + 10L_{40}(x + y)(x^2 + y^2)}{(x^2 + y^2)^5}.$$

□

12. QUINTIC QUASIHOMOGENOUS POLYNOMIAL PLANAR VECTOR
FIELD WITH NON-DEGENERATE CENTER

We shall study the quintic polynomial vector field

$$\begin{aligned}
 \dot{x} &= -y + L_{50}x^5 + L_{05}y^5 + L_{23}x^2y^3 + L_{32}x^3y^2 \\
 &\quad + L_{41}x^4y + L_{14}xy^4 := -y + X, \\
 \dot{y} &= x + K_{50}x^5 + K_{05}y^5 + K_{23}x^2y^3 + K_{32}x^3y^2 \\
 &\quad + K_{41}x^4y + K_{14}xy^4 := x + Y,
 \end{aligned}
 \tag{96}$$

Proposition 77. *Differential system (96) can be rewrite as follows*

$$\begin{aligned}
 \dot{x} &= -\frac{\partial H_2 + H_6}{\partial y} - yg_4, \\
 \dot{y} &= \frac{\partial H_2 + H_6}{\partial x} + xg_4,
 \end{aligned}$$

where $H_6 = H_6(x, y)$ and $g_4 = G_4(x, y)$ are convenient homogenous polynomial of degree six and four respectively, if and only if

$$\int_0^{2\pi} \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) |_{x=\cos t, y=\sin t} dt = K_{41} + K_{23} + L_{23} + L_{14} + 5(K_{05} + L_{50}) = 0,
 \tag{97}$$

where X and Y are polynomials given by the formula (96)

Proof. Indeed, under the condition (97) the polynomial H_6 and g_4 are

$$\begin{aligned}
 H_6 &= -\frac{(K_{32} + L_{23} + 2K_{14} + 2L_{41} + 2K_{50})}{12}y^6 - L_{50}x^5y \\
 &\quad - \frac{L_{41} + K_{50}}{2}x^4y^2 - \frac{L_{23} + 2L_{41} + 2K_{50} + K_{32}}{4}x^2y^4 \\
 &\quad + \frac{(K_{23} + L_{14} + 5K_{05})}{3}x^3y^3 + \Lambda(x^2 + y^2)^3 + K_{05}xy^5, \\
 g_4 &= \frac{(L_{23} + L_{23} + 2L_{41} + 2K_{14} + K_{32})}{2}y^4 - (K_{23} + L_{14} + L_{32} + 5K_{05})x^3y \\
 &\quad - (L_{14} + 5K_{05})xy^3 + (2L_{41} + K_{32})x^2y^2 + 2(K_{50} - 6\Lambda)(x^2 + y^2)^2
 \end{aligned}$$

□

Proposition 78. *Quintic differential system*

$$\begin{aligned}\dot{x} &= -y + L_{50}x^5 + L_{41}x^4y - \frac{1}{\lambda}(3\lambda K_{05} + L_{50}(2 - 3\lambda))y^3x^2 + \frac{K_{05}(5\lambda - 2)}{\lambda}xy^4 \\ &\quad + L_{23}x^2y^3 + \frac{(3\lambda - 1)}{2\lambda^3}(\lambda^2L_{23} - 2\Lambda(2\lambda^2 - 3\lambda + 1) + \lambda(1 - 2\lambda)L_{41})y^5, \\ \dot{y} &= x - \frac{2 - 5\lambda}{\lambda}L_{50}x^4y - \left(\frac{(3\lambda - 2)K_{05}}{\lambda} - 3L_{50}\right)x^2y^3 - K_{05}y^5 \\ &\quad + \frac{1}{2\lambda^3}(\lambda^2(\lambda - 1) + \Lambda(6\lambda^2 - 8\lambda + 2) + \lambda(3\lambda - 1)L_{41})xy^4.\end{aligned}$$

is Poincaré–Liapunov integrable.

Proof. Indeed, the function $F = (1 + (1 - \lambda)g_4)H_2^{(\lambda-1)/\lambda}$ if $\lambda \in \mathbb{R} \setminus \{0, 1\}$ where

$$\begin{aligned}g_4 &= -\frac{2\Lambda}{\lambda}x^4 + \frac{2L_{50}}{\lambda}x^3y - \frac{2K_{05}}{\lambda}xy^3 \\ &\quad - \frac{2(\lambda - 1)\Lambda + \lambda L_{41}}{\lambda^2}x^2y^2 + \frac{1}{2\lambda^3}(\lambda^2L_{23} - 2(2\lambda^2 - 3\lambda + 1)\Lambda + \lambda(-2\lambda)L_{41})y^4,\end{aligned}$$

and $F = H_2e^{-\tilde{g}}$ where

$$\tilde{g} = g|_{\lambda=1} = -2\Lambda x^4 + 2L_{50}x^3y + L_{41}x^2y^2 - 2K_{05}xy^3 + \frac{L_{23} - L_{41}}{2}y^4,$$

are first integrals. \square

Corollary 79. *The quintic quasi-homogenous differential system*

$$\begin{aligned}\dot{x} &= -y + x(L_{50}x^4 + L_{41}x^3y + L_{23}xy^3 + K_{05}y^4 - 3(K_{05} + L_{50})x^2y^2), \\ \dot{y} &= x + y(L_{50}x^4 + L_{41}x^3y + L_{23}xy^3 + K_{05}y^4 - 3(K_{05} + L_{50})x^2y^2),\end{aligned}$$

, ,

has an uniformly isochronous center at the origin and admits the following rational first integral

$$F = \frac{(4L_{50}x^3y - 2L_{41}x^2y^2 - 4K_{05}xy^3 + (L_{41} + L_{23})y^4 - 4\Lambda(x^2 + y^2)^2)^2}{(x^2 + y^2)^3}.$$

Proof. Follows from Proposition 78 with $\lambda = 1/3$ and in view of the Corollary 18. \square

Remark 80. In the paper [23] the following result was proposed.

Theorem 81. *The quintic differential system*

$$\begin{aligned}\dot{x} &= -y := P, \\ \dot{y} &= x + K_{50}x^5 + K_{05}y^5 + K_{23}x^2y^3 + K_{32}x^3y^2 + K_{41}x^4y + K_{14}xy^4 := Q,\end{aligned}$$

admits a center at the origin if and only if the coefficients A_{kn} such that

$$P + iQ = iz + \sum_{k+n=5} A_{kn} z^k \bar{z}^n,$$

are real, i.e.,

$$(98) \quad \begin{aligned} K_{05} - K_{23} + K_{41} &= 0, \\ 5K_{05} - K_{23} - 3K_{41} &= 0, \\ 5K_{05} + K_{23} + K_{41} &= 0, \end{aligned}$$

It is easy to show that the unique quintic polynomial vector field for which (98) holds is

$$\dot{x} = -y, \quad \dot{y} = x + K_{50}x^5 + K_{32}x^3y^2 + K_{14}xy^4,$$

which is a particular case of the reversible system (22) with $m = 5$.

The following counterexample show that this theorem is only sufficient condition.

The non-reversible quintic system

$$(99) \quad \dot{x} = -y, \quad \dot{y} = x - 5x^4y + y^5$$

has a center at the origin .

13. ANALYTIC PLANAR VECTOR FIELD WITH NON-DEGENERATE CENTER

In this section we study the inverse problem of the center for the case when the vector field is analytic.

We shall study the analytic planar differential system

$$(100) \quad \dot{x} = -\frac{\partial H}{\partial y} - yg, \quad \dot{y} = \frac{\partial H}{\partial x} + xg,$$

where $g = g(x, y)$ is an analytic function and $H = \frac{\lambda}{2}(x^2 + y^2) + f(x, y)$, and $f = f(x, y)$ is a real analytic functions in an open neighborhood of O whose Taylor expansions at O do not contain constant, linear and quadratic terms.

Proposition 82. *Differential system (100) is Hamiltonian if and only if $g = g(x^2 + y^2)$.*

Proof. From divergence condition for (100) follows that

$$x \frac{\partial g}{\partial y} - y \frac{\partial g}{\partial x} = 0,$$

consequently $g = g(x^2 + y^2)$. The Hamiltonian is

$$H = V + \frac{1}{2} \int g(x^2 + y^2) d(x^2 + y^2).$$

□

Now we study the analytic planar differential system

$$(101) \quad \dot{x} = \sigma x - (\tilde{\nu} + 1)y = -y + X, \quad \dot{y} = \sigma y + (\tilde{\nu} + 1)x = x + Y,$$

where $\sigma = \sigma(x, y)$ and $\tilde{\nu} = \tilde{\nu}(x, y)$ are analytic functions. In polar coordinates this differential system becomes

$$(102) \quad \dot{r} = r\sigma, \quad \dot{\vartheta} = \tilde{\nu} + 1 := \nu.$$

Corollary 83. *Differential system (102) is Hamiltonian if and only if*

$$(103) \quad \frac{\partial \varrho \nu}{\partial \vartheta} + \frac{\partial \varrho r \sigma}{\partial r} = 0.$$

where ϱ is the integrating factor. The Hamiltonian is

$$H = \int \varrho (\nu dr - r \sigma d\vartheta).$$

Proof. Indeed, from the relations

$$\frac{\partial H}{\partial \vartheta} = -\varrho r \sigma, \quad \frac{\partial H}{\partial r} = \varrho \nu,$$

and in view of the compatibility conditions we obtain (103). □

We study the particular case when $\sigma = \frac{\partial A}{\partial \theta}$ and $\varrho = \varrho(r)$, where $A = A(r, \theta)$. Under these conditions (103) becomes

$$\frac{\partial}{\partial \theta} \left(\varrho(r) \nu + \frac{\partial}{\partial r} (r \varrho(r) A) \right) = 0,$$

thus

$$\nu = -\frac{1}{\varrho(r)} \left(\frac{\partial}{\partial r} (r \varrho(r) A) + \tau(r) \right)$$

where $\tau = \tau(r)$ is an arbitrary function. The first integral is

$$H = r \varrho(r) A(r, \theta) + q(r), \quad q'(r) = \tau(r).$$

Differential system (102) in this case becomes

$$(104) \quad \dot{r} = \frac{1}{\varrho} \frac{\partial}{\partial \theta} (r \varrho A + q(r)), \quad \dot{\vartheta} = -\frac{1}{\varrho} \frac{\partial}{\partial r} (r \varrho A + q(r)).$$

Now we study the generalized weak condition of the center.

Corollary 84. *If differential system (101) satisfy the conditions*

$$(105) \quad \begin{aligned} (x^2 + y^2) \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) &= k(xX + yY), \quad k \in \mathbb{R} \setminus \{0\}, \\ \sigma &= \frac{\partial A}{\partial \theta} \end{aligned}$$

then is Poincaré-Liapunov integrable with the first integral

$$H = r^{2-k} A(r, \vartheta) + q(r),$$

where $q = q(r)$ is an arbitrary function.

Proof. In view of the relations

$$\begin{aligned} \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} &= x \frac{\partial \sigma}{\partial x} + y \frac{\partial \sigma}{\partial y} + x \frac{\partial \nu}{\partial y} - y \frac{\partial \nu}{\partial x} + 2\sigma, \\ xX + yY &= \kappa(x^2 + y^2)\sigma, \end{aligned}$$

from (105) we have

$$x \frac{\partial \sigma}{\partial x} + y \frac{\partial \sigma}{\partial y} + x \frac{\partial \nu}{\partial y} - y \frac{\partial \nu}{\partial x} = (\kappa - 2)\sigma,$$

which in polar coordinates becomes

$$r \frac{\partial \sigma}{\partial r} + (2 - k)\sigma + \frac{\partial \nu}{\partial \theta} = 0,$$

consequently

$$\frac{\partial}{\partial \theta} \left(r \frac{\partial A}{\partial r} + (2 - k)A + \nu \right) = 0,$$

thus

$$\nu = -r^{k-1} \left(\frac{\partial r^{2-k} A}{\partial r} + p(r) \right),$$

where $p = p(r)$ is an arbitrary function. By introducing the function $q = q(r)$ such that $q'(r) = p(r)$ we finally obtain that the given differential system in polar coordinates becomes

$$\dot{r} = r^{k-1} \frac{\partial}{\partial \theta} (r^{2-k} A + q(r)), \quad \dot{\vartheta} = -r^{k-1} \frac{\partial}{\partial r} (r^{2-k} A + q(r)).$$

We observe that this system can be obtained from (104) with $\varrho = r^{1-k}$. \square

Corollary 85. *Differential system (104) has an isochronous point at the origin if*

$$(106) \quad A(r, \theta) = \frac{\Psi(\vartheta) - \int \varrho(r) dr - q(r)}{r \varrho(r)}$$

Proof. Indeed, if (106) holds then (102) becomes

$$(107) \quad \dot{r} = \frac{1}{\varrho(r)} \frac{\partial \Psi(\vartheta)}{\partial \theta}, \quad \dot{\vartheta} = 1.$$

The Hamiltonian function in this case takes the form

$$H = r \varrho A(r, \vartheta) + q(r) = \Psi(\vartheta) - \int \varrho(r) dr.$$

□

Remark 86. *The following remarks are related with differential equations (102).*

- (i) *In [15] was studied the existence of a uniformly isochronous polynomial system has the form*

$$\dot{x} = -y + xG(x, y)\Omega(x^2 + y^2), \quad \dot{y} = x + yG(x, y)\Omega(x^2 + y^2),$$

where $G = G(x, y)$ is a homogenous polynomial in x and y of degree k and $\int_0^{2\pi} G(\cos t, \sin t) dt = 0$. Clearly this system is a particular case of system (107).

- (ii) *The center problem for the system (102) with $\sigma = \frac{\partial A}{\partial \theta}$ was study in [6]. The following corollary was proved (see Corollary 2.21)*

Corollary 87. *Let now $H(x, y) \in \mathbb{C}[x, y]$ be a homogeneous polynomial. For any holomorphic functions P_1, P_2 defined in an open neighborhood of $0 \in \mathbb{C}$ we define $A(x, y) := P_1(H(x, y))$, and $B(x, y) := P_2(H(x, y))$. Then the vector field*

$$(108) \quad \begin{aligned} \dot{x} &= \left(x \frac{\partial A}{\partial y} - y \frac{\partial A}{\partial x} \right) x - (B + 1)y, \\ \dot{y} &= \left(x \frac{\partial A}{\partial y} - y \frac{\partial A}{\partial x} \right) y + (B + 1)x, \end{aligned}$$

determines a center.

Clearly system (38) is a particular case of (108).

- (iii) *Characterization of isochronous point for planar system was study in particular in [1].*

14. THE CENTER-FOCI PROBLEM

As we observe in the previous section the center- foci problem consists into distinguish when the origin of (3) is a center or foci.

We need the following definitions:

By using the polar coordinates $x = r \cos \theta$, $y = r \sin \theta$ in (3) and denoting by $r(t, r_0, \theta_0)$, $\theta(t, r_0, \theta_0)$ the solution of the (3) such that $r|_{t=t_0} = r_0$, $\theta|_{t=t_0} = \theta_0$. We say that the origin is stable focus if there exists $\delta > 0$ such that for $0 < r_0 < \delta$ and $\theta_0 \in \mathbb{R}$, we have that

$$\lim_{t \rightarrow \infty} r(t, r_0, \theta_0) = 0, \quad \text{and} \quad \lim_{t \rightarrow \infty} |\theta(t, r_0, \theta_0)| = \infty,$$

and unstable if

$$\lim_{t \rightarrow -\infty} r(t, r_0, \theta_0) = 0, \quad \text{and} \quad \lim_{t \rightarrow -\infty} |\theta(t, r_0, \theta_0)| = \infty.$$

To solve the center- foci problem Poincaré and Liapunov developed the method which is given in the following theorems (see for instance [25, 19]) .

Theorem 88. *For the system (3) there exists a formal power series*

$$(109) \quad V(x, y) = \sum_{j=2}^{\infty} H_j(x, y), \quad H_2(x, y) = \frac{1}{2} (x^2 + y^2),$$

where $H_j(x, y) = H_j$ is a homogenous function of degree j . such that

$$(110) \quad \begin{aligned} \frac{dV}{dt} &= \left(x + \frac{\partial H_3}{\partial x} + \frac{\partial H_4}{\partial x} + \dots \right) \left(-\lambda y + \sum_{j=2}^m X_j(x, y) \right) \\ &\quad + \left(y + \frac{\partial H_3}{\partial y} + \frac{\partial H_4}{\partial y} + \dots \right) \left(\lambda x + \sum_{j=2}^m Y_j(x, y) \right) \\ &= \sum_{j=0}^{\infty} V_j(x^2 + y^2)^{j+1}, \end{aligned}$$

where V_j are constants called Poincaré-Liapunov constants.

The main objective of the next section is to study the following inverse problem in ordinary differential equations (see for instance [31]).

Problem 89. *Determine the analytic planar vector fields*

$$\mathcal{X} = (-y + \sum_{j=2}^{\infty} X_j(x, y)) \frac{\partial}{\partial x} + (x + \sum_{j=2}^{\infty} Y_j(x, y)) \frac{\partial}{\partial y},$$

where X_j and Y_j for $j \geq 2$ are unknown homogenous polynomial of degree j , in such a way that (110) holds.

Problem 90. Study the Problem 89 when \mathcal{X} is polynomial of degree m , i.e.

$$\mathcal{X} = (-y + \sum_{j=2}^m X_j(x, y)) \frac{\partial}{\partial x} + (x + \sum_{j=2}^m Y_j(x, y)) \frac{\partial}{\partial y},$$

From (109) and (110) if $\lambda = 1$ we obtain the matrix equation

$$A \Psi = B + C,$$

where A , Ψ , B and C are matrix such that

$$\begin{pmatrix} \partial_x H_2 & \partial_y H_2 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \partial_x H_3 & \partial_y H_3 & \partial_x H_2 & \partial_y H_2 & 0 & 0 & 0 & \dots & 0 \\ \partial_x H_4 & \partial_y H_4 & \partial_x H_3 & \partial_y H_3 & \partial_x H_2 & \partial_y H_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ \partial_x H_m & \partial_y H_m & \partial_x H_{m-1} & \partial_y H_{m-1} & \vdots & \vdots & \vdots & \partial_y H_2 & \partial_x H_2 \\ \partial_x H_{m+1} & \partial_y H_{m+1} & \partial_x H_m & \partial_y H_m & \vdots & \vdots & \vdots & \partial_y H_3 & \partial_x H_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$\Psi = (X_2, Y_2, X_3, Y_3, \dots, X_m, Y_m, 0, \dots, 0, \dots)^T$, for polynomial vectot fields

$\Psi = (X_2, Y_2, X_3, Y_3, \dots, X_m, Y_m, \dots, \dots)^T$, for analytic vectot fields

$B = (\{H_3, H_2\}, \{H_4, H_2\}, \{H_5, H_2\}, \dots, \{H_m, H_2\}, \dots)^T$,

$C = (V_1(x^2 + y^2)^2, 0, V_2(x^2 + y^2)^3, 0, V_4(x^2 + y^2)^4, 0, \dots, V_m(x^2 + y^2)^m, \dots)^T$,

where $\{f, g\} := f_x g_y - f_y g_x$, $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$.

15. QUASIHOMOGENOUS POLYNOMIAL DIFFERENTIAL SYSTEM OF DEGREE EVEN WITH A FOCI AT THE ORIGIN

The problem which we study in this and the following section is the following.

Problem 91. Determine the quasihomogenous polynomial vector field

$$(111) \quad \dot{x} = -y + X_m(x, y), \quad \dot{y} = x + Y_m(x, y),$$

where $X_m = X_m(x, y)$, and $Y_m = Y_m(x, y)$ are homogenous polynomial of degree m in such a way that

$$\begin{aligned}
 \frac{dV}{dt} &= \left(x + \frac{\partial H_3}{\partial x} + \frac{\partial H_4}{\partial x} + \dots \right) (-y + X_m(x, y)) \\
 &+ \left(y + \frac{\partial H_3}{\partial y} + \frac{\partial H_4}{\partial y} + \dots \right) (x + Y_m(x, y)) \\
 (112) \quad &= \sum_{j=0}^{\infty} V_j (x^2 + y^2)^{j+1},
 \end{aligned}$$

In this section we study the case when the degree of the vector field is even.

Proposition 92. *Polynomial vector field (111) of degree $m = 2k - 2$ for which (112) holds is*

$$\begin{aligned}
 \dot{x} &= -y - \frac{\partial H_{2k-1}}{\partial y} - yg_{2k-3}, \\
 \dot{y} &= x + \frac{\partial H_{2k-1}}{\partial x} + xg_{2k-3},
 \end{aligned}
 \quad (113)$$

where $g_{2k-3} = g_{2k-3}(x, y)$, is a homogenous polynomial of degree $2k - 3$, and

$$\begin{aligned}
 (114) \quad H_4 &= \kappa_2 H_2^2, \quad H_6 = \kappa_3 H_2^3, \dots, H_{2k-2} = \kappa_{k-1} H_2^{k-1}, \\
 H_3 &= H_5 = H_7 = \dots = H_{2k-3} = 0, \\
 V_1 &= V_2, \dots = V_{k-2} = 0, \\
 g_{2k-1} \{H_2, H_{2j+1-2k}\} &+ \{H_{2j+1-2k}, H_{2k+1}\} + \{H_2, H_{2j}\} = V_j (x^2 + y^2)^{j+1}, \\
 g_{2k-1} \{H_{2j+2-2k}, H_2\} &+ \{H_{2j+2-2k}, H_{2k+1}\} + \{H_2, H_{2j+1}\} = 0,
 \end{aligned}$$

for $j \geq k - 1$.

Proof. From (113) follows that if $m = 2k - 2$ then

$$\begin{aligned}
\{H_3, H_2\} &= 0, \\
\{H_4, H_2\} &= V_1(x^2 + y^2)^2, \\
\{H_5, H_2\} &= 0, \\
\{H_6, H_2\} &= V_2(x^2 + y^2)^3, \\
&\vdots \qquad \qquad \vdots \qquad \qquad \vdots \qquad \qquad \vdots \\
\{H_{2k-3}, H_2\} &= 0, \\
\{H_{2k-2}, H_2\} &= V_{k-2}(x^2 + y^2)^{k-1}, \\
xX_{2k-2} + yY_{2k-2} + \{H_{2k-1}, H_2\} &= 0, \\
\frac{\partial H_3}{\partial x}X_{2k-2} + \frac{\partial H_3}{\partial y}Y_{2k-2} + \{H_{2k}, H_2\} &= V_{k-1}(x^2 + y^2)^k, \\
&\vdots \qquad \qquad \vdots \qquad \qquad \vdots \qquad \qquad \vdots.
\end{aligned}$$

Hence by considering that that

$$\int_0^{2\pi} \{f, H_2\}|_{x=\cos t, y=\sin t} dt = 0,$$

for arbitrary derivable function f , we obtain that $V_j = 0$ for $j = 2, \dots, k - 2$.

In view of the relations

$$\{H_2, H_j\} = 0 \iff f = \begin{cases} 0, & \text{if } j = 2n + 1, \\ \kappa_j(x^2 + y^2)^j, & \text{if } j = 2n + 1. \end{cases}$$

we have that $H_{2j} = \kappa_j H_2^j$, and $H_{2j-1} = 0$ for $j = 1, 2, \dots, k - 2$.

Finally from the equation $xX_{2k-2} + yY_{2k-2} + \{H_{2k-1}, H_2\} = 0$, follows that

$$X_{2k-2} = -\frac{\partial H_{2k-1}}{\partial y} - yg_{2k-3}, \quad Y_{2k-2} = \frac{\partial H_{2k-1}}{\partial x} - xg_{2k-3}.$$

By inserting in the remain equations we obtain the last two systems of first order partial differential equations, where $g_{2k-3} = g_{2k-3}(x, y)$ is an arbitrary polynomial of degree $2k - 3$. Thus we obtain the proof of the proposition. \square

Corollary 93. *Any quasihomogenous differential equations of degree $m = 2k - 2$ can be rewrite as (111).*

16. QUASIHOMOGENOUS POLYNOMIAL DIFFERENTIAL SYSTEM OF
DEGREE ODD WITH A FOCI AT THE ORIGIN

Proposition 94. *Polynomial vector field (111) of degree $m = 2k - 1$ for which (112) holds is*

$$\begin{aligned}\dot{x} &= -y - \frac{\partial H_{2k}}{\partial y} - yg_{2k-2} + V_{k-1}(x^2 + y^2)^{k-1}x, \\ \dot{y} &= x + \frac{\partial H_{2k}}{\partial x} - xg_{2k-2} + V_{k-1}(x^2 + y^2)^{k-1}y,\end{aligned}$$

and

(115)

$$\begin{aligned}H_4 &= \kappa_2 H_2^2, \quad H_6 = \kappa_3 H_2^3, \dots, H_{2k-2} = \kappa_{k-1} H_2^{k-1}, \\ H_{2l+1} &= 0, \quad \text{for } l = 1, 2, \dots \\ 0 &= V_1 = V_2, \dots = V_{k-1}, \\ 0 &= g_{2k-2} \{H_{2m+1-2k}, H_2\} + \{H_{2m+1-2k}, H_{2k}\} + \{H_{2m-1}, H_2\}, \\ V_{m-1}(x^2 + y^2)^{m-1} &= g_{2k-2} \{H_{2m-2k}, H_2\} + \{H_{2m-2k}, H_{2k}\} + \{H_{2m-2}, H_2\} \\ &\quad + (2m - 2k)V_k(x^2 + y^2)^{k-1}H_{2m-2k},\end{aligned}$$

for $m \geq k$, where $g_{2k-2} = g_{2k-2}(x, y)$, is a homogenous polynomial of degree $2k - 2$,

Proof. From (113) if $m = 2k - 1$ follows that

$$\begin{aligned}\{H_3, H_2\} &= 0, \\ \vdots & \quad \quad \quad \vdots \quad \quad \quad \vdots \\ \{H_{2k-3}, H_2\} &= 0, \\ \{H_{2k-2}, H_2\} &= V_{k-2}(x^2 + y^2)^{k-1}, \\ xX_{2k-1} + yY_{2k-1} + \{H_{2k}, H_2\} &= V_k(x^2 + y^2)^k, \\ \frac{\partial H_3}{\partial x}X_{2k-1} + \frac{\partial H_3}{\partial y}Y_{2k-1} + \{H_{2k+1}, H_2\} &= 0, \\ \frac{\partial H_4}{\partial x}X_{2k-1} + \frac{\partial H_4}{\partial y}Y_{2k-1} + \{H_{2k+2}, H_2\} &= V_{k+1}(x^2 + y^2)^{k+1}, \\ \vdots & \quad \quad \quad \vdots \quad \quad \quad \vdots \\ \frac{\partial H_{2l+1}}{\partial x}X_{2k-1} + \frac{\partial H_{2l+1}}{\partial y}Y_{2k-1} + \{H_{2k+2l-1}, H_2\} &= 0\end{aligned}$$

Thus after some computations we prove the proposition. \square

Corollary 95. *Liapunov constants for the vector field (113) and (??) can be computing as follows*

(116)

$$V_j = \frac{1}{2\pi} \int_0^{2\pi} (g_{2k-1}\{H_2, H_{2j+3-2k}\} + \{H_{2j+4-2k}, H_{2k}\})|_{x=\cos t, y=\sin t} dt$$

where $j \geq k$, if $m = 2k$ and

$$V_j = \frac{1}{2\pi} \int_0^{2\pi} (g_{2k-2}\{H_2, H_{j-k+1}\} + \{H_{2(j-k)+1}, H_{2k}\} + 2kV_{j-1}H_{2k})|_{x=\cos t, y=\sin t} dt$$

where $j \geq k$ if $m = 2k - 1$.

Proof. Follows from (114) and by (115) by considering that

$$\int_0^{2\pi} \{f, H_2\}|_{x=\cos t, y=\sin t} dt = 0,$$

for arbitrary derivable function f . □

Example 96. *We shall consider Bautin quadratic system (see [3])*

$$(117) \quad \begin{aligned} \dot{x} &= -y - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\ \dot{y} &= x + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2, \end{aligned}$$

By comparing (113) and (117) we obtain that

$$\begin{aligned} \frac{\partial H_3}{\partial x} &= (B + 2\lambda_3 + \lambda_4)yx + (\lambda_2 + A)x^2 - \lambda_4 y^2, \\ \frac{\partial H_3}{\partial y} &= (A - 2\lambda_2 - \lambda_5)yx + \lambda_3 x^2 + (B - \lambda_6)y^2, \\ g_1 &= Ax + By. \end{aligned}$$

From the compatibility conditions and after integration we deduce that

$$\begin{aligned} g_1 &= \lambda_5 x - \lambda_4 y, \\ H_3 &= \frac{1}{3}(\lambda_2 + \lambda_5)x^3 + \lambda_3 x^2 y - \frac{1}{3}(\lambda_2 + \lambda_6)y^3. \end{aligned}$$

The Liapunov constants in this case we calculated by using formula (116). It is possible to show that

$$V_1 = \frac{1}{2\pi} \int_0^{2\pi} g_1\{H_2, H_3\}|_{x=\cos t, y=\sin t} dt = \frac{1}{8}\lambda_5(\lambda_3 - \lambda_6).$$

Example 97. *We shall consider quasi-homogenous cubic system (70). After some computations we can prove that in this case the function*

H_4 , g_2 and the Liapunov constant V_2 in this case are

$$\begin{aligned} H_4 &= (V_2 - A)x^3y + \frac{M}{2}x^2y^2 - \frac{D}{4}y^4 + \frac{M+B+K}{4}x^4 + \left(\frac{N}{4} - V_2\right)x^3y \\ &\quad + \frac{\Lambda}{4}(x^2 + y^2)^2, \\ g_2 &= -(M+B)x^2 - (4V_2 - 3A - L)xy + \Lambda(x^2 + y^2), \\ V_2 &= \frac{1}{8}(3(A+N) + L + C). \end{aligned}$$

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